MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1932 DECEMBER 9

Dr. H. Knox-Shaw, President, in the Chair.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Stanley M. Ballance, A.R.P.S., F.R.Met.Soc., 3 Westfield Avenue, Harpenden, Herts (proposed by R. A. Fisher);

James Blackhall, Woodlands, Hythe, Hants (proposed by J. McD. Field); and

F. O'B. Ellison, Principal of the Government Medical College, Colombo, Ceylon (proposed by W. F. A. Ellison).

Fifty-eight presents were announced as having been received since the last meeting, including:—

L. Edwards, The Spangled Heavens: an Introduction to Astronomy;
W. M. Feldman, Rabbinical Mathematics and Astronomy.

Mr. H. W. Newton and Mr. W. M. Witchell were appointed Auditors of the Treasurer's accounts for 1932.

HORNSBY'S OBSERVATIONS OF MERCURY, 1774–1798.

J. Jackson.

(Communicated by the Astronomer Royal)

When the Radcliffe Observatory, Oxford, was founded in 1772 it was equipped with a transit instrument, two mural quadrants and a zenith sector. With these instruments the first Radcliffe Observer, Rev. Thomas Hornsby, and his two immediate successors Robertson and Rigaud, observed assiduously from 1774 till 1838, but the observations were left in manuscript unreduced. During this period the observations included about 130,000 transits and 60,000 zenith distances.

Although astronomers were aware of this great mass of observations it was only recently that a serious attempt was made to deal with it. With the help of grants from the Royal Society a reduction of Hornsby's observations between 1774 and 1798 was undertaken, and the results obtained have been published in a volume recently issued from the Radcliffe Observatory.* This volume is based on a reduction of the following observations:—

	Right Ascensions, 1774–84 and 1790–98	Declination, 1774–98
Standard stars	33,000	10,000
Sun	2,200	3,000
Moon	600	600
Planets	2,300	2,300

The standard stars are fully discussed in the volume as also are the Sun observations. For the latter a complete ephemeris of the Sun was constructed from Newcomb's tables with modern accuracy, so that it was possible to deduce not only corrections to the elements of the Earth's orbit, but also to the position of the equinox and the system of declinations. To avoid further delay in publishing the results obtained, it was decided to print in the volume the observed positions of the Moon and planets and to leave a discussion of the observations to a later date.

The observations of Mercury are specially important. The probable error of a single star observation was found to be only about $\pm 0^s.05$ and $\pm 1''.1$, and it was presumed that the observations of Mercury would be of a similar degree of accuracy. There are 103 complete observations between 1774 and 1778, 106 between 1779 and 1784, and 140 between 1790 and 1798. These observations should, therefore, afford a good determination of the orbital elements of Mercury about 150 years ago. In particular they will, in conjunction with modern observations, form a strong check on the motion of the perihelion of the planet. As is well known, Newcomb introduced

* *The Observations of the Rev. Thomas Hornsby, D.D., made with the transit instrument and quadrant at the Radcliffe Observatory, Oxford*, reduced by H. Knox-Shaw, J. Jackson and W. H. Robinson.

into his tables a value 43″·37 a century in excess of that computed from the perturbations of the planet in accordance with Newton's law of gravitation, but in close agreement with the motion later calculated by Einstein from the general theory of relativity.

For comparison it may be mentioned that Newcomb's tables were based on 5421 observations made between 1750 and 1890, but of these the only meridian observations before 1791 were those made at Greenwich, the total number from 1774–1790 being 46. The weight of the few meridian observations made by Bradley at a still earlier date is very small.

Observations of Mercury depend equally on the positions of the planet and of the Earth, and are suitable for the determination of the orbital elements of the Earth round the Sun. The mean position of the planet is approximately that of the Sun, and in fact the positions of the planet are computed by means of its elongation from the Sun. These observations, therefore, are suitable for determining the positions of the equinox and equator point amongst the stars a century and a half ago. Moreover, any want of regularity in the Earth's rotation will produce an apparent displacement of the Sun in longitude, and so give a check on the rather uncertain direct determinations of the Sun's longitude at the end of the eighteenth century. It might appear at first sight that Mercury as a fast-moving planet would itself afford a better check on the rotation of the Earth, but the planet is generally observed fairly near to elongation when a displacement along its orbit does not affect its position as seen from the Earth.

The heaviest part of the work was the calculation of tabular values of Mercury with all the accuracy of Newcomb's tables. There were too few observations to justify the computation of a continuous ephemeris which could be used for controlling the accuracy of the work. The *Nautical Almanac* gave the positions of Mercury only to minutes of arc at intervals of several days and could not be used for checking purposes. To avoid mistakes, the work was therefore done in duplicate where control formulæ could not be used. The first thing to do was to calculate the Greenwich Mean Time of observation. Fortunately, the aberration time could be obtained directly, to about 1ˢ, from modern almanacs, as Mercury returns to nearly the same position relative to the Sun and Earth at intervals of 92 years and $2\frac{1}{2}$ days—a period very nearly equal to 290 synodic revolutions of Mercury and approximately to 92 sidereal revolutions of the Earth. Half this period might of course be used, but the longer period brings us to a time when almanacs of approximately the present accuracy were available. The ephemeris was, therefore, calculated directly for the time of observation, corrected for the aberration time. A daily ephemeris of the Sun for every Greenwich Mean Noon had already been made and could be used for reduction of the heliocentric positions of Mercury to geocentric.

The observations were reduced on Boss's system in both right ascension and declination to local apparent place. When the ephemeris was constructed the parallax was applied, and the only real uncertainty in the discussion involved the defective illumination. The semi-diameter of Mercury at distance unity is 3″·34. On many days the effect of phase is only a small

fraction of this, and often it is too small to be noticed in the telescope on account of bad definition. Still a great many of the observations of Mercury were made near the time of maximum elongation when the phase is marked, and some were even made when the planet was near inferior conjunction, although these are not so numerous as those near superior conjunction. In all Hornsby's observations the centre is given as the point observed. Nowadays, when an observation is made of a planet which does not present a circular disc the observer either observes the limb or the "centre of illumination." In his discussion of the observations of Mercury Newcomb took this interpretation of the meaning of the planet's centre and applied a correction equal to the semi-diameter multiplied by $\frac{1}{12}(1-\cos i)(5+\cos i)$, where i is the angle between the Earth and the Sun as seen from the planet. This formula has been constructed so as to give zero when the planet is in superior conjunction and to fit an assumption that when the thickness of the crescent approaches zero the point observed is two-thirds of the way from the centre to the limb. Newcomb, however, found unexpected residuals in the observations depending on the position of the planet in its orbit relative to the Earth. He wrote: "The most natural conclusion is that the reduction from the limb of the planet or the observed centre of light to the true centre was too small by an amount which at the mean distance of the Sun must have been nearly or quite a second of arc. The adopted semi-diameter $3''{\cdot}4$ seems so well established both by micrometric measures and heliometer measures during transits of Mercury that such a correction of the diameter seems inadmissible."

In the case of Mercury the phase correction is of the order of or less than the probable error of observation, and it is impossible from the observations to decide whether the phase correction as computed from Newcomb's formula should be applied or not. The comparison of Hornsby's observations of Venus with theory has now been completed, and with this planet the phase correction may amount to as much as $10''$, and in some cases it clearly should not be applied. In fact, the residuals seem to indicate that contrary to all practice recognised at present, Hornsby sometimes saw the disc of Venus partly illuminated and actually estimated at the telescope the geometrical centre of the complete disc. In such a case the question naturally arises as to the wisdom of applying any phase corrections in the case of Mercury or Venus. When the phase corrections were small it would hardly be possible for the observer to do anything but observe the centre of illumination. One might be tempted to suggest that the phase corrections should be applied when they are small but not when they are large, but in the case of Mercury they are never very large, and in any case the point of separation would be indefinite. The important point is the effect which the application or non-application of the corrections will have in the final results. Clearly the principal effect of the application of too large a correction for phase would be to make the size of Mercury's orbit too large—but the size must be extremely well known from Kepler's third law, as the period of revolution is known with great accuracy. If the observations were equally balanced on the east and west sides of the Sun, it would appear that

it would make little difference on most of the elements whether the phase correction was applied or not, but it has appeared best in the present paper to make the analysis with and without the corrections applied.

The original observations have been published in the Radcliffe volume already referred to, pp. 112–115. The table at the end of this paper gives the residuals without the application of any correction for defective illumination and also the corrections to be applied according to Newcomb's formula. Some of the residuals are rather large, but it has been decided to reject none of them except the R.A. for 1778 October 31, for which the observed R.A. appears to be 2^s too small. The observed R.A. for 1791 July 15 has been decreased by 10^s. A scrutiny of the original reductions of the calculations of tabular places has led to the application of the following corrections to the published declinations:—

		°	′	″		°	′	″
1779 May 30	For	+13	35	32·6	read	+13	35	43·9
1793 July 7	,,	+22	5	11·8	,,	+22	5	16·2
1794 June 18	,,	+25	3	20·8	,,	+25	3	7·6
1795 Aug. 22	,,	+17	11	18·4	,,	+17	11	28·4

In analysing the residuals the first question which arises is the number of unknowns to be introduced. Theoretically, corrections to all the elements of both the planet's and the Earth's orbit should be introduced. Newcomb also included a correction to the mass of Venus, but it does not seem advisable to introduce it now when more modern observations are available for the same purpose. Two additional corrections, however, should be introduced, namely, corrections to the right ascensions and declinations of Boss's system of standard stars used in the determination of the instrumental constants.

For the right ascensions it was decided to include a correction to only one element of the Sun's orbit (or rather to the Earth's orbit round the Sun), viz. the longitude, and to incorporate this with the constant correction to the system of right ascensions. This meant that we took the cosine of the obliquity as unity and considered that the eccentricity of the solar orbit was so well known that any error in the adopted value would not appreciably affect normal equations based on observations extending over several years. For the planet it was decided to neglect corrections to the inclination and node which could produce little effect on the right ascensions and to solve only for corrections to the longitude, eccentricity and longitude of perihelion. Equations of condition were therefore calculated in the form

$$x+\frac{d\alpha}{dl}\Delta l+\frac{d\alpha}{de}\Delta e+\frac{d\alpha}{d\varpi}\Delta\varpi=\Delta\alpha,$$

where Δl, Δe, $\Delta\varpi$ are corrections to the longitude, eccentricity and longitude of perihelion of Mercury, and x is a constant to be later equated to $\Delta l''-E$ where $\Delta l''$ is the correction to the Sun's longitude and E is the correction to the adopted system of right ascensions. The equations of condition in declination were calculated in the form

$$-y+\frac{d\delta}{dl''}\Delta l''+\frac{d\delta}{d\epsilon}\Delta\epsilon=\Delta\delta,$$

where y is the correction to Boss's system of declinations and $\Delta\epsilon$ the correction to the obliquity of the ecliptic.

The differential coefficients for the above equation were computed by the formulæ and tables given by Newcomb in vol. ii of the *Astronomical Papers of the American Ephemeris*. It might be noted here that several of the formulæ on p. 17 of that work are not in the form used in the tables, and the formula for $\frac{d\delta}{dl''}$ is erroneous, having the factor $\sin e \cos L$ repeated in two different forms. Only one alteration was introduced in forming the normal equations. Newcomb's tables do not give $\frac{d\alpha}{dl}$ but $\frac{d\alpha}{dg}$, where g is the mean anomaly. Hence, if Newcomb's formulæ were used the equation of condition for $\Delta\alpha$ would be

$$x+\frac{d\alpha}{dg}\Delta g+\frac{d\alpha}{de}\Delta e+\frac{d\alpha}{d\varpi}\Delta\varpi=\Delta\alpha.$$

But the coefficients of Δg and $\Delta\varpi$ in this equation are so nearly equal that extra arithmetic would be required in forming and solving the normal equations if Δg and $\Delta\varpi$ were retained as the unknowns.

The latter equation was therefore transformed to

$$x+\frac{d\alpha}{dg}\Delta l+\frac{d\alpha}{de}\Delta e+10\left(\frac{d\alpha}{d\varpi}-\frac{d\alpha}{dg}\right)\frac{\Delta\varpi}{10}=\Delta\alpha,$$

since $\Delta l=\Delta g+\Delta\varpi$.

The Right Ascensions.—In order to have a good check on the accuracy of the results the normal equations were formed and solved separately in three periods, 1774–78, 1779–84, 1790–94. Residuals from observations not more than six days apart were meaned before forming equations of condition. All observations were given equal weight. The normal equations for the three periods were as follows, the correction for defective illumination being given on the extreme right:—

102	x	$+11{\cdot}88\Delta l$	$+9{\cdot}98\Delta e$	$-13{\cdot}02\frac{\Delta\varpi}{10}$	$=+$	$''$ 80·16	$-$ $''$ 6·67
+ 11·88	+ 3·20	+ 0·94	− 2·27		+	5·94,	− 0·43
+ 9·98	+ 0·94	+11·07	− 1·16		−	8·79,	+ 6·82
− 13·02	− 2·27	− 1·16	+48·76		+	23·50,	− 1·41

106	x	$+13{\cdot}27\Delta l$	$-2{\cdot}20\Delta e$	$-20{\cdot}09\frac{\Delta\varpi}{10}$	$=+$	$''$ 131·27,	$-$ $''$ 8·52
+ 13·27	+ 4·45	− 1·38	− 4·34		+	14·81,	+ 1·22
− 2·20	− 1·38	+13·30	+ 1·79		−	19·10,	+ 8·51
− 20·09	− 4·34	+ 1·79	+52·92		−	31·81,	+ 5·94

140	x	$+20{\cdot}59\Delta l$	$+17{\cdot}33\Delta e$	$-26{\cdot}45\frac{\Delta\varpi}{10}$	$=+$	$''$ 279·28,	$+$ $''$ 28·92
+ 20·59	+ 5·39	+ 3·73	− 7·57		+	38·87,	+ 0·76
+ 17·33	+ 3·73	+16·36	+ 1·23		+	36·84,	+ 1·86
− 26·45	− 7·57	+ 1·23	+59·50		−	66·89,	+ 1·67

The solution of these equations gives the following results, when corrections for defective illumination are not applied:—

	1774–78	1779–84	1790–98	1774–98
	″	″	″	″
x	+1·25	+1·33	+2·13	+1·55
Δl	−1·79	−1·20	−1·81	−0·64
Δe	−1·69	−1·32	+0·44	−0·69
$\Delta\varpi$	+6·92	−1·49	−4·18	+0·60

and when corrections for defective illumination are applied:—

	1774–78	1779–84	1790–98	1774–98
	″	″	″	″
x	+1·07	+1·14	+2·55	+1·57
Δl	−1·52	−0·02	−3·23	−0·64
Δe	−0·94	−0·60	+0·43	−0·28
$\Delta\varpi$	+6·46	−0·37	−3·81	+1·00

It will be noticed that that application of corrections for defective illumination does not make any appreciable difference in the agreement of the values of x and $\Delta\varpi$ for the three periods, and that it improves it for Δe and reduces it for Δl. For the first two periods the observations are more or less equally divided between east and west of the Sun, so that on the whole the phase corrections balance, but for the third period there are considerably more observations east of the Sun, the algebraic sum of the phase corrections being 28″·92. The variation of the corrections in the various periods are larger and deserve closer examination. x is made up of an equinox correction which should be nearly constant, and of a correction to the Sun's longitude which may be different at different times if the angular rotation of the Earth during the interval under consideration was different from the average. An examination of the residuals for the years 1790–98 seems to indicate that all the years agree in giving the larger value of x, and that it is not derived from discordant observations during a few years. As already indicated, the longitude of Mercury is not very well determined, and in fact its probable error is such that the values for the different periods must be considered as in rather good agreement before the corrections are applied. The application of the corrections has improved the corrections for the eccentricity, although the disagreement still exceeds the probable errors of the separate determinations. Attention is specially drawn to the variation of $\Delta\varpi$. As the eccentricity is about 0·2 and heliocentric distance about 0·4 the maximum effect of an error $\Delta\varpi$ on the geocentric place of the planet near elongation is only about $0{\cdot}08\Delta\varpi$. On account of the importance of this element in connection with the theory of relativity the residuals for the separate years have been examined. The large value for the first period is mainly produced by the residuals in 1777, and especially the two large residuals on May 4 and 17 of that year. If these were rejected the value of $\Delta\varpi$ would be reduced by 2″·2. There is no other year in the whole series which appears to affect the result considerably, although the rejection of a residual of 5″ or over might occasionally influence the result by a considerable fraction of its probable error.

A comparison of the yearly residuals with the computed values gives the probable error of a single observation for each of the three periods as $\pm 1''\cdot3$, $\pm 0''\cdot8$, $\pm 1''\cdot1$ or for the whole series of observations $\pm 1''\cdot1$. Reduced to time this is $\pm 0^s\cdot073$, which may be compared with $\pm 0^s\cdot052$ found for the star observations.

The probable errors of the various determinations are therefore:—

	1774–78	1779–84	1790–98
	"	"	"
x	$\pm 0\cdot2$	$\pm 0\cdot15$	$\pm 0\cdot14$
Δl	$\pm 1\cdot0$	$\pm 0\cdot5$	$\pm 0\cdot8$
Δe	$\pm 0\cdot4$	$\pm 0\cdot2$	$\pm 0\cdot3$
$\Delta\omega$	$\pm 1\cdot9$	$\pm 1\cdot2$	$\pm 1\cdot6$

The Declinations.—The normal equations were formed as indicated above for the three periods 1774–78, 1779–84, 1790–98. The second value on the right is the correction to be applied for defective illumination.

$$\left.\begin{array}{rrrrrr} 103 & y+ & 6\cdot10\Delta l'' & -20\cdot54\delta\epsilon = & +62''\cdot38, & +9''\cdot97 \\ +6\cdot10 & +7\cdot22 & +0\cdot19 & & +11\cdot56, & -0\cdot99 \\ -20\cdot54 & +0\cdot19 & +42\cdot52 & & +17\cdot39, & -12\cdot95 \end{array}\right\}$$

$$\left.\begin{array}{rrrrrr} 106 & y- & 1\cdot98\Delta l'' & -16\cdot70\delta\epsilon = & +27''\cdot42, & +12''\cdot00 \\ -1\cdot98 & +8\cdot93 & +1\cdot08 & & +0\cdot27, & -1\cdot36 \\ -16\cdot70 & +1\cdot08 & +34\cdot13 & & +21\cdot21, & -14\cdot89 \end{array}\right\}$$

$$\left.\begin{array}{rrrrrr} 140 & y+ & 9\cdot30\Delta l'' & -35\cdot73\delta\epsilon = & +58''\cdot34, & +0''\cdot95 \\ +9\cdot30 & +10\cdot30 & -2\cdot73 & & +10\cdot82, & +4\cdot67 \\ -35\cdot73 & -2\cdot73 & +46\cdot67 & & +22\cdot07, & -10\cdot12 \end{array}\right\}$$

The solutions of these equations give the following values when corrections for defective illumination are not applied:—

	1774–78	1779–84	1790–98	1774–98
	"	"	"	"
y	$+0\cdot69$	$+0\cdot39$	$+0\cdot62$	$+0\cdot58$
$\Delta l''$	$+1\cdot00$	$+0\cdot02$	$+0\cdot75$	$+0\cdot61$
$\Delta\epsilon$	$+0\cdot74$	$+0\cdot81$	$+0\cdot99$	$+0\cdot84$

and when the corrections for defective illumination are applied:—

	1774–78	1779–84	1790–98	1774–98
	"	"	"	"
y	$+0\cdot75$	$+0\cdot43$	$+0\cdot53$	$+0\cdot58$
$\Delta l''$	$+0\cdot82$	$-0\cdot07$	$+1\cdot22$	$+0\cdot68$
$\Delta\epsilon$	$+0\cdot46$	$+0\cdot40$	$+0\cdot73$	$+0\cdot53$

Again the application of corrections for defective illumination makes very little difference, except that the correction to the obliquity of the ecliptic has been reduced.

The values of the corrections in the last period have been considerably influenced by several large positive residuals in the year 1793, and these large residuals influence the determination of the probable error which comes out for the three periods as $\pm 1''\cdot2$, $\pm 0''\cdot8$, $\pm 1''\cdot7$ for a single observation as compared with $\pm 1''\cdot1$ for the star observations. If the latter value

is adopted the probable error of the various quantities from all three periods combined is

$$y \pm 0''{\cdot}06 \qquad \Delta l'' \pm 0''{\cdot}2 \qquad \Delta\epsilon \pm 0''{\cdot}11.$$

In the following paragraphs, in quoting the results deduced from the observations of Mercury, the values obtained by applying the corrections for defective illumination are used unless otherwise stated.

The Longitude of the Sun and the Equinox Correction.—These elements are important in connection with irregularities in the rotation of the Earth and the secular term in the Sun's longitude. The longitude of the Sun from the true equinox can really only be determined from declination observations, but its longitude from the equinox of a system of clock stars can be determined more directly from observations of right ascension, although these are affected by difference in the personal equation of the observer for the Sun and stars.

The corrections to Newcomb's tables derived from the observed right ascensions of the Sun have been given by Ross in *A.J.*, **29**, 153. The only other observations before 1800 discussed by Newcomb when constructing his tables were those made at Palermo from 1791 onwards, but these were rejected. In the Radcliffe volume recently published the corrections to the Sun's longitude depending on the right ascensions are not given, but they can easily be recovered from the published data and they are given below as also the values from the R.A. observations of Mercury discussed in this paper. The results are as follows:—

Greenwich		Radcliffe Observatory, Oxford			
		Sun		Mercury	
	″		″		″
1750–62	−0·63				
65–71	−0·28				
72–78	−0·35	1775–79	+0·09	1775–78	+1·07
79–85	+1·94	80–84	+1·42	79–84	+1·14
86–92	+0·69	90–93	+2·48		
93–97	+1·13	94–98	+1·75	90–98	+2·55
98–02	+0·3				

The corrections to the Sun's longitude deduced from the observed declinations are given by Newcomb in *Astronomical Constants*, pp. 30–31, and in the Radcliffe volume on Hornsby's observations, p. 101. Newcomb's values refer to Leverrier's tables of the Sun and have been reduced to his own tables by means of the data given by Ross in the paper already quoted.

Greenwich		Radcliffe Observatory, Oxford			
		Sun		Mercury	
	″		″		″
1753–57	−0·18				
58–62	+0·64				
65–70	−0·88				
71–78	−0·61	1774–79	−0·65	1774–78	+0·82
79–85	+0·2	80–84	−0·49	79–84	−0·07
86–91	−0·34	85–89	−0·27		
92–97	+0·38	90–93	+0·10	90–98	+1·22
98–03	−0·3	94–98	+1·11		

The determinations from the declination observations show much greater regularity than those from the right ascensions, although the coefficients in the normal equations and the superior type of instrument used would have made us anticipate that the smoothest results would have been secured from the right ascensions. Apparently the observations in R.A. are considerably affected by varying personal equation in observations of the Sun or stars.

The additive correction to Boss's right ascensions (correction to Newcomb's equinox) derived from the Radcliffe observations are as follows:—

	Sun		Mercury
	″		″
1774–79	−0·74	1774–78	−0·25
80–84	−1·91	79–84	−1·21
90–93	−2·38		
94–98	−0·64	90–98	−1·33

The Sun observations indicate the mean correction to the equinox about 1785 as $-1''\cdot42$ or $-0^s\cdot09$, while the Mercury observations indicate a correction of $-0''\cdot93$ or $-0^s\cdot06$. The general trend of corrections to the longitude of the Sun as derived from Newcomb's tables and referred to the true equinox may be taken as

	″
1775	−0·25
1785	+0·20
1795	+0·65

Rotation of the Earth.—The astronomical data available for testing irregular rotation of the Earth have been collected and carefully discussed by Professor de Sitter in *B.A.N.*, **124**. He deduced the following corrections to Newcomb's tabular longitudes:—

$$\text{Sun} \quad +1''\cdot89 + 1''\cdot41T + 1''\cdot80S + 0\cdot094B',$$
$$\text{Mercury} \quad +7\cdot65 + 7\cdot13T + 6\cdot43S + 0\cdot388B',$$

where T is measured in centuries from 1900, $S = T^2 + 1\cdot33T - 0\cdot26$ and B' is the fluctuation of the Moon. The coefficients of B' in the above equations correspond to the ratio of the mean motions relative to that of the Moon multiplied by 1·25. The following table gives the values of B' together with the calculated corrections to the longitudes, the part due to B' being given separately from the secular part:—

	B'	Sun		Mercury	
	″	″	″	″	″
1770	+13·6	−0·48 + 1·28 =	+0·80	−3·54 + 5·28 =	+1·74
1780	15·4	−0·55 + 1·45	+0·90	−3·58 + 5·98	+2·40
1790	14·7	−0·58 + 1·38	+0·80	−3·49 + 5·70	+2·21
1800	+13·1	−0·58 + 1·23	+0·65	−3·27 + 5·08	+1·81

The values of the Sun's longitude derived from Hornsby's observations indicate an increasing correction to Newcomb's tables instead of a decreasing one, while the observed corrections to Newcomb's tabular longitude for Mercury, though not very well determined, are all negative. At the time

covered by the observations B' had a maximum value. Clearly, for this epoch a better fit would be obtained by decreasing the factor of B' to the value obtained directly from the relative mean motions and by increasing the secular term.

The Obliquity of the Ecliptic.—The average correction to the obliquity of the ecliptic is considerable, $+0''.53$, and a still larger value would have been derived if corrections for defective illumination had not been applied. The declination of the Sun indicated a smaller correction of $+0''.21$, while a determination of very small weight from the right ascensions was $-0''.54$. The latter may be dismissed as unreliable. As the scale of the quadrant was derived from Boss's declinations this result would indicate either a true correction to the obliquity or a correction to the arc between declination $+20°$ and $-20°$ in Boss's system. It should be pointed out that recent determinations indicate possible corrections to Newcomb's obliquity, *e.g.* for epoch 1785.

Spencer Jones:	Occultations	$+0''.50$
,,	Greenwich observations of Sun 1821–1927	$+0''.28$
Brown:	,, ,, ,, Moon	$+0''.37$

Also it might be noted that Newcomb obtained larger corrections to the obliquity from planets than from the Sun. On p. 96 of *Astronomical Constants* he gave the corrections to Leverrier's obliquity. These values reduced to his own obliquity are

From observations of the Sun	$-0''.15$
Mercury	$+0''.28$
Venus	$+0''.28$
Mars	$+0''.40$

The Equator Point.—The correction to the Boss declinations for 1785 has a mean value of $+0''.58$ whether corrections for defective illumination are applied or not. Recent determinations have all been negative, varying from $-0''.09$ by Spencer Jones to $-0''.59$ by Eichelberger. On p. 101 of the Radcliffe volume the following corrections were given:—

1774–79	1780–84	1785–89	1790–93	1794–98
$-0''.49$	$-0''.53$	$-0''.10$	$-0''.66$	$+0''.29$

But the declination observations of the Sun from which these values were obtained were simultaneously solved for corrections to the eccentricity and the longitude of perihelion of the Sun. If the much better values of these quantities derived from the R.A.'s are substituted, or if the normal equations in R.A. and Dec. are added and solved together, systematically different and more reliable values are obtained as follows:—

1774–79	1780–84	1785–89	1790–93	1794–98
$+0''.12$	$-0''.16$	$+0''.02$	$-0''.30$	$+0''.46$

The mean of the values is $+0''.03$ instead of $-0''.30$. This paper supports Boss's declinations against suggested negative corrections near the equator about 1785.

The Perihelion of Mercury.—If we take the three determinations of the correction to the perihelion and weight them according to their probable errors we get the value for 1785 as $0''.0 \pm 0''.9$. The discordance of the three determinations indicates a probable error considerably greater than this, and from a general consideration I would put the probable error at twice this value. In any case this determination must be easily the best which can be obtained from observations before 1800. In fact, the discovery of the discordance between theory and observation was made by Leverrier not from meridian observations but from observations of transits of the planet across the Sun. In his confirmation of Leverrier's discovery Newcomb also gave great weight to the observations of transits. It has recently been pointed out, however, that if there are irregularities in the rotation of the Earth of the magnitude suggested by observations of the Moon, there is great risk in using the transits for this purpose.* The transit observations give equations involving several elements and of these the longitude of the planet in its orbit plays an important part. It has already been pointed out that in meridian observations of the planet near elongation an error in the planet's heliocentric longitude does not seriously influence its geocentric position, but in the case of transits of the planet across the Sun its heliocentric longitude is all important, and variations in its error from one transit to another would have large effects on the deduced position of the perihelion.

In *A.J.*, **40**, 50, Morgan has given the corrections to the perihelion of Mercury derived from modern observations as follows:—

			No. of Observations
		"	
Washington	1902·0	+2·5	257
,,	1908·6	+3·3	513
Cape	1909·5	+0·4	248
,,	1914·5	+2·4	221
Washington	1921·8	+1·8	258
Mean	1911	$+2''.30 \pm 0''.3$	1497

By combining Hornsby's observations with modern ones, we therefore get a correction to the centennial motion of the perihelion of Mercury amounting to $+1''.8 \pm 1''.5$ if we make a liberal estimate of the probable error. Newcomb estimated the mean error of the determination of the centennial motion of $e\Delta\varpi$ as $\pm 0''.40$, giving the probable error in the motion of the perihelion as $\pm 1''.3$ per century. If we combine the two results with equal weight we get the correction to the motion of the perihelion of Mercury given in the tables as $+0''.9 \pm 1''.0$ per century.

Newcomb considered that his tables gave a motion of the perihelion of Mercury $43''.37$ per century in excess of the value derived from the Newtonian Theory. There have been several revisions of this value. The Hornsby observations give a correction of over $-1''$ to Newcomb's equinox at 1785, and this would indicate an increase in the constant of

* Von Gleich, *A.N.*, **241**, 105; Fotheringham, *M.N.*, **91**, 1001.

precession. If, following de Sitter,* we accept the constant of precession as 5024″·90 instead of 5023″·71 for 1850·0, the departure of the tables from Newtonian theory is reduced to 42″·18 per century. Although Newcomb gave the probable error of his theoretical motion of the perihelion of Mercury as only ± 0″·5, Dolittle † has shown on the basis of the theory revised by himself and Hill that Newcomb's theoretical variation is 3″ too large, and that the discordance between theory and observation must be increased by this amount. There is also some uncertainty in the masses of the planets which further complicates the problem from the theoretical standpoint. Morgan considered that the tables departed from theory by 44″·3 per century.

If we use de Sitter's value of the difference between Newcomb's tables and theory we get the difference between the observed motion and Newtonian theory as 43″·1 ± 1″·0 per century, in close agreement with the value computed from the theory of general relativity, viz. 42″·9.

I have to acknowledge the assistance of Mr. Bawtree, who made all the duplicate calculations for the tabular positions of Mercury for the three years 1774–77. A few of the reductions from heliocentric to geocentric place were made with the help of the generous grant by the Government Grant Committee of the Royal Society for the reduction of Hornsby's Observations.

Residuals Uncorrected for Phase, and the Corrections for Phase

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1774	s	″	s	″
Mar. 15	+ ·09	−2·6	+·04	+0·3
20	− ·06	−2·2	+·07	+0·6
Apr. 30	+ ·50	−0·4	−·15	−1·2
July 14	+ ·12	+2·3	+·09	−0·4
16	+ ·12	+1·6	+·09	−0·5
24	·00	−0·4	+·12	−0·8
30	− ·02	+0·2	+·14	−1·2
Aug. 7	− ·23	−2·8	+·18	−1·7
Sept. 7	+ ·11	−7·2	−·07	+0·4
8	− ·16	−1·8	−·06	+0·4
Oct. 24	− ·12	−4·4	+·01	−0·1
26	+ ·25	−1·8	+·01	−0·1
1775				
Jan. 5	+ ·07	+2·1	−·03	+0·2
Feb. 26	+ ·10	−4·1	+·03	+0·3
28	+ ·13	−3·3	+·04	+0·3
Apr. 19	+ ·04	−3·0	−·11	−0·9
26	+ ·26	−0·9	−·08	−0·7
27	+ ·03	−1·9	−·08	−0·6
28	+ ·24	−3·9	−·07	−0·6
30	+ ·17	+4·9	−·07	−0·6
May 1	+ ·16	−2·2	−·07	−0·5
3	+ ·19	−0·8	−·06	−0·5
13	+ ·09	+3·7	−·03	−0·2

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1775	s	″	s	″
June 4	− ·01	+0·7	+·01	+0·3
5	− ·02	−1·8	+·01	+0·3
16	− ·06	−0·8	+·06	0·0
17	− ·43	−3·2	+·06	0·0
18	− ·05	−0·6	+·06	0·0
26	+ ·02	+1·3	+·10	−0·2
Aug. 24	− ·06	−3·7	−·06	+0·3
Sept. 24	+ ·06	−2·1	·00	0·0
26	− ·01	−3·3	+·01	−0·1
Oct. 5	− ·15	−3·7	+·01	−0·1
20	+ ·31	−3·8	+·04	−0·2
21	+ ·02	+0·4	+·04	−0·2
25	− ·02	0·0	+·06	−0·3
26	− ·02	+0·8	+·06	−0·3
Dec. 9	+ ·22	+5·0	−·05	+0·3
10	− ·10	+3·0	−·05	+0·2
11	− ·09	+2·2	−·05	+0·2
18	+ ·16	+2·4	−·03	+0·1
25	− ·01	−2·0	−·01	0·0
1776				
Feb. 12	+ ·20	−4·8	+·04	+0·2
16	− ·04	−1·2	+·07	+0·5
Mar. 25	+ ·28	+2·3	−·13	−0·9
Apr. 9	+ ·13	+0·3	−·07	−0·6

* *M.N.*, **76**, 728.

† *Transactions of the American Phil. Soc.*, New Series, **22**, 186.

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1776	s	″	s	″
Apr. 27	+ ·16	+0·5	−·02	−0·2
May 28	+ ·01	+1·6	+·05	+0·1
June 4	+ ·09	−0·8	+·09	+0·1
9	+ ·15	+0·3	+·12	−0·1
14	+ ·03	−2·1	+·14	−0·2
July 25	+ ·05	+0·5	−·18	−0·1
29	+ ·10	+2·4	−·14	+0·1
31	− ·05	+4·6	−·12	+0·1
Aug. 7	− ·14	+2·7	−·06	+0·2
16	+ ·01	+1·1	·00	0·0
21	− ·01	−1·9	−·01	+0·2
Sept. 1	+ ·25	−3·4	·00	−0·1
6	+ ·15	−1·7	+·01	−0·1
9	+ ·06	+2·5	+·01	−0·1
10	+ ·12	+3·8	+·01	−0·1
14	+ ·13	−3·7	+·01	−0·1
Oct. 18	+ ·03	−5·0	+·12	−0·7
Nov. 16	+ ·03	−2·4	−·09	+0·6
24	+ ·07	+2·9	−·04	+0·3
25	+ ·10	−0·6	−·04	+0·2
1777				
Jan. 7	− ·03	0·0	·00	0·0
8	+ ·25	+1·6	·00	0·0
Feb. 2	+ ·02	−3·7	+·10	+0·6
Mar. 21	+ ·02	−1·0	−·07	−0·5
24	+ ·13	+0·7	−·06	−0·4
25	+ ·01	−2·4	−·05	−0·4
May 4	− ·27	+8·7	+·01	+0·1
17	− ·48	+6·1	+·08	+0·2
29	− ·03	−0·3	+·17	+0·1
July 16	+ ·04	+2·4	−·11	−0·8
Sept. 10	+ ·10	−5·4	+·04	−0·3
11	+ ·18	−6·3	+·04	−0·3
14	+ ·07	−3·2	+·05	−0·3
16	+ ·11	−1·4	+·05	−0·4
17	+ ·24	−4·5	+·05	−0·4
19	+ ·08	−3·5	+·06	−0·5
20	+ ·11	+0·3	+·06	−0·5
Nov. 4	+ ·13	−4·8	−·05	+0·4
6	+ ·18	+3·0	−·05	+0·4
16	+ ·16	+1·2	−·01	+0·1
1778				
Feb. 15	+ ·02	−1·3	−·14	−0·4
Mar. 26	+ ·19	−0·4	−·01	−0·1
Apr. 25	− ·10	+1·9	+·04	+0·2
26	− ·14	−2·0	+·05	+0·3
June 20	+ ·19	0·0	−·17	−0·8
July 5	+ ·04	+0·2	−·07	−0·1
7	+ ·09	+0·8	−·06	−0·1
9	− ·05	+1·6	−·05	0·0
13	− ·03	+0·1	−·03	0·0
15	+ ·18	+2·2	−·02	0·0
Aug. 7	+ ·05	−2·6	+·02	−0·1

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1778	s	″	s	″
Aug. 12	− ·15	+1·3	+·02	−0·1
Oct. 12	+ ·12	−0·9	−·12	+0·9
16	+ ·05	−2·5	−·08	+0·6
17	+ ·09	−1·5	−·07	+0·6
25	+ ·03	−2·4	−·03	+0·3
31	−1·81	−3·5	−·01	+0·1
1779				
Jan. 28	+ ·05	−0·7	−·15	0·0
Mar. 10	+ ·15	+0·2	−·01	−0·1
29	+ ·09	+1·1	·00	0·0
30	+ ·32	+1·8	·00	0·0
31	+ ·15	+3·2	+·01	+0·1
Apr. 2	+ ·09	+0·9	+·01	+0·1
3	+ ·05	+2·2	+·02	+0·1
4	− ·03	−0·5	+·02	+0·1
5	− ·04	−0·8	+·02	+0·2
9	− ·07	+3·4	+·04	+0·2
14	+ ·02	+1·0	+·07	+0·4
May 30	+ ·22	−4·8	−·17	−1·2
June 2	+ ·39	+2·6	−·16	−1·0
3	+ ·07	−1·1	−·15	−1·0
5	+ ·06	+2·4	−·14	−0·9
Sept. 26	+ ·04	−2·6	−·12	+0·8
30	+ ·18	−0·7	−·08	+0·6
Oct. 2	+ ·16	+0·3	−·06	+0·5
4	+ ·15	+0·2	−·05	+0·4
7	+ ·10	−1·1	−·03	+0·3
16	− ·10	−1·6	−·01	+0·1
17	− ·10	−2·5	−·01	0·0
19	+ ·06	−1·6	·00	0·0
21	− ·12	−0·6	·00	0·0
Nov. 7	+ ·16	−4·3	·00	0·0
1780				
Feb. 22	+ ·24	+3·9	−·01	−0·1
25	+ ·15	−3·1	−·01	0·0
Mar. 22	+ ·08	−1·4	+·03	+0·2
23	+ ·26	+0·3	+·03	+0·2
24	+ ·05	+1·0	+·04	+0·3
25	+ ·17	−2·6	+·05	+0·3
May 28	+ ·18	+2·9	−·08	−0·5
June 7	− ·03	−1·5	−·04	−0·1
29	− ·02	−1·0	+·01	0·0
30	+ ·09	−0·6	+·01	0·0
July 27	− ·04	+1·6	+·09	−0·5
28	+ ·02	−0·9	+·09	−0·6
Aug. 3	+ ·11	−2·3	+·11	−0·8
4	+ ·10	+1·2	+·11	−0·8
7	− ·01	+2·0	+·13	−1·0
11	− ·04	−2·3	+·14	−1·3
13	− ·32	+1·1	+·15	−1·4
Sept. 5	+ ·35	+2·8	−·17	+0·7
11	+ ·07	−3·8	−·11	+0·7
18	+ ·07	−1·0	−·05	+0·4

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1780	s	″	s	″
Sept. 20	+ ·19	−2·3	−·03	+0·3
Oct. 1	+ ·19	+2·8	−·01	+0·1
4	+ ·03	+1·8	·00	0·0
20	+ ·15	−0·2	·00	0·0
Nov. 15	+ ·08	−2·4	+·03	−0·1
1781				
Jan. 18	+ ·16	−2·0	−·03	0·0
24	+ ·16	+0·7	−·02	0·0
26	− ·01	+1·3	−·01	0·0
Mar. 18	+ ·05	−5·9	+·11	+1·0
Apr. 18	+ ·27	+2·7	−·17	−1·3
May 8	+ ·14	−3·2	−·08	−0·6
9	+ ·18	−2·5	−·08	−0·6
12	+ ·07	+2·3	−·06	−0·5
26	+ ·04	+0·8	−·02	−0·1
27	+ ·29	+2·1	−·02	−0·1
29	+ ·02	+2·3	−·01	0·0
31	+ ·08	+0·3	−·01	0·0
June 1	·00	+0·6	·00	0·0
July 16	+ ·09	+0·1	+·13	−0·7
Aug. 30	+ ·09	−0·4	−·08	+0·4
Sept. 3	+ ·31	+2·2	−·04	+0·3
6	+ ·10	+1·3	−·03	+0·2
9	+ ·07	+1·2	−·01	+0·1
Oct. 16	+ ·18	−0·8	+·01	−0·1
26	− ·19	−1·4	+·03	−0·2
1782				
Jan. 4	+ ·01	−2·1	−·02	0·0
June 3	+ ·02	−1·3	+·03	+0·1
16	− ·12	+1·1	+·09	−0·1
1783				
Jan. 22	+ ·07	−2·2	+·04	0·0
29	+ ·08	−1·2	+·02	0·0
Feb. 14	+ ·03	+1·6	+·11	+0·9
Mar. 15	+ ·17	−3·7	−·15	−0·8
16	+ ·21	+0·5	−·14	−0·8
18	+ ·31	−2·7	−·13	−0·8
20	+ ·25	−1·8	−·12	−0·7
Apr. 3	+ ·12	−1·8	−·06	−0·5
5	+ ·10	−0·3	−·06	−0·5
9	+ ·06	−2·3	−·05	−0·4
27	+ ·12	+0·8	−·01	−0·1
28	+ ·06	−0·7	−·01	−0·1
29	+ ·05	+3·0	−·01	0·0
30	+ ·01	−0·6	·00	0·0
May 3	− ·04	−1·1	·00	0·0
10	+ ·19	+1·3	·00	0·0
12	+ ·02	+2·2	+·01	0·0
June 8	− ·04	−1·7	+·15	−0·1
Sept. 23	+ ·01	+0·5	+·04	−0·3
26	+ ·06	−0·9	+·05	−0·3
27	− ·08	−0·7	+·05	−0·3
28	− ·06	−2·6	+·05	−0·4

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1783	s	″	s	″
Sept. 30	− ·18	−1·3	+·06	−0·4
Oct. 3	− ·06	+1·2	+·07	−0·5
5	+ ·10	+4·6	+·08	−0·5
Nov. 19	+ ·25	+1·1	−·03	+0·2
1784				
Jan. 6	+ ·21	−1·6	+·01	0·0
24	+ ·07	−0·3	+·07	+0·3
25	+ ·11	−2·4	+·08	+0·4
Sept. 12	+ ·16	−0·4	+·06	−0·5
Oct. 26	+ ·15	−0·6	−·07	+0·5
Dec. 17	+ ·16	−5·1	+·01	0·0
22	− ·13	−0·2	+·01	0·0
1790				
Jan. 29	+ ·28	−2·2	+·04	+0·2
30	+ ·27	−0·5	+·05	+0·3
Feb. 1	+ ·08	−8·8	+·07	+0·4
6	+ ·15	−4·6	+·11	+0·7
Mar. 31	+ ·12	−3·2	−·05	−0·4
Apr. 3	+ ·20	−1·3	−·04	−0·3
15	+ ·17	−0·3	−·01	+0·3
17	+ ·12	−0·3	−·01	+0·2
May 3	+ ·23	+0·2	·00	+0·2
16	− ·06	−1·4	+·06	+0·2
23	− ·06	−0·7	+·10	+0·1
Sept. 6	+ ·13	0·0	+·03	−0·2
17	+ ·08	−1·3	+·04	−0·3
18	+ ·10	−3·3	+·05	−0·4
Nov. 14	+ ·10	−0·4	−·02	+0·2
15	+ ·15	+2·4	−·02	+0·2
19	+ ·24	−0·3	−·01	+0·1
23	+ ·22	+2·2	−·01	+0·1
26	+ ·16	+3·7	−·01	0·0
27	+ ·05	+1·3	·00	0·0
1791				
Mar. 2	+ ·09	−3·2	−·08	−0·4
Apr. 23	+ ·06	+1·1	+·02	+0·1
25	+ ·12	+3·4	+·03	+0·2
June 27	+ ·23	−0·2	−·15	−0·1
28	+ ·16	+2·0	−·14	−0·1
July 3	+ ·34	−8·6	−·11	−0·2
14	+ ·13	+1·7	−·03	0·0
15	+ ·19	+2·0	−·03	0·0
16	+ ·11	−2·4	−·02	+0·1
Aug. 19	+ ·07	−3·2	+·03	−0·2
20	+ ·12	−0·9	+·03	−0·2
23	+ ·11	−2·4	+·04	−0·2
Sept. 10	+ ·10	−0·5	+·08	−0·7
12	·00	+0·9	+·09	−0·8
14	+ ·18	−0·8	+·10	−0·8
15	+ ·17	+2·0	+·11	−0·9
16	+ ·05	−2·7	+·11	−0·9
Oct. 16	+ ·38	−0·8	−·10	+0·8
23	+ ·31	+1·7	−·05	+0·4

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1791	s	″	s	″
Oct. 24	+ ·36	−3·9	−·04	+0·3
27	+ ·15	+1·3	−·03	+0·3
Nov. 4	+ ·11	−1·5	−·01	+0·1
1792				
Mar. 7	+ ·16	−0·4	−·02	−0·1
11	+ ·08	−1·2	−·01	−0·1
June 8	+ ·20	+4·0	−·14	−0·8
July 16	+ ·13	−2·7	·00	0·0
28	+ ·11	+0·1	+·03	−0·1
30	+ ·11	−0·1	+·03	−0·2
Aug. 2	+ ·13	+0·8	+·04	−0·2
5	+ ·20	−1·1	+·04	−0·3
6	+ ·16	−3·1	+·05	−0·3
7	+ ·10	+0·9	+·05	−0·3
19	− ·01	−0·6	+·08	−0·6
23	+ ·05	−1·2	+·09	−0·8
1793				
Jan. 26	+ ·16	−1·0	−·06	−0·1
27	− ·08	+1·7	−·06	−0·1
Mar. 29	+ ·17	0·0	+·06	+0·4
30	− ·03	−2·0	+·06	+0·4
Apr. 5	+ ·17	+2·7	+·11	+0·9
6	+ ·08	−0·5	+·12	+0·9
7	− ·08	+2·9	+·13	+1·0
9	+ ·01	−1·2	+·15	+1·1
10	+ ·06	+2·6	+·15	+1·2
June 6	+ ·41	+1·4	−·05	−0·3
9	− ·01	+3·6	−·04	−0·2
July 6	− ·07	+7·0	+·02	0·0
7	− ·05	0·0	+·02	0·0
9	+ ·05	−2·4	+·03	−0·1
10	+ ·24	+0·8	+·03	−0·1
12	+ ·10	+7·5	+·04	−0·1
13	+ ·26	+1·0	+·04	−0·1
14	+ ·04	+0·6	+·04	−0·1
16	+ ·12	+2·8	+·05	−0·2
Aug. 1	+ ·35	−2·9	+·09	−0·5
Sept. 25	+ ·08	−4·6	−·02	+0·2
26	+ ·22	+3·0	−·02	+0·2
27	+ ·23	+3·5	−·02	+0·2
Oct. 30	− ·09	+0·3	+·01	0·0
1794				
Apr. 25	+ ·42	+2·7	−·15	−1·2
May 3	+ ·11	−1·1	−·11	−0·9
10	+ ·25	+4·2	−·08	−0·7
15	+ ·06	−2·3	−·07	−0·5
June 17	− ·13	−2·0	+·01	0·0
18	+ ·11	+1·6	+·02	0·0
July 2	+ ·11	−2·1	+·06	−0·1
8	+ ·13	−4·9	+·09	−0·3
13	+ ·15	+1·4	+·10	−0·5
14	+ ·11	+3·2	+·10	−0·5
Aug. 30	+ ·47	+2·4	−·10	+0·5
Sept. 14	+ ·05	−3·8	−·04	+0·4

	Obs. – Comp.		Phase Corr.	
	R.A.	Dec.	R.A.	Dec.
1794	s	″	s	″
Oct. 12	− ·01	+2·4	+·01	0·0
13	+ ·27	+0·6	+·01	0·0
21	+ ·12	−4·5	+·02	−0·1
1795				
Mar. 1	− ·01	−2·7	+·08	+0·6
May 9	+ ·34	+1·2	−·03	−0·2
17	+ ·27	−0·4	−·01	−0·1
18	+ ·10	+0·5	−·01	−0·1
20	− ·03	−2·5	·00	0·0
Aug. 15	+ ·41	−1·1	−·10	+0·3
22	+ ·16	−4·5	−·05	+0·2
25	+ ·23	−0·3	−·03	+0·2
29	+ ·24	−0·3	−·01	+0·1
Sept. 13	+ ·39	+1·2	·00	0·0
17	− ·01	−1·7	·00	0·0
18	− ·11	−2·3	·00	0·0
21	+ ·02	−5·1	+·01	0·0
22	+ ·23	−2·4	+·01	0·0
25	·00	+0·1	+·01	−0·1
28	+ ·23	−3·9	+·01	−0·1
1796				
Feb. 4	+ ·13	+0·4	+·03	+0·1
14	+ ·15	−3·4	+·09	+0·6
May 17	+ ·15	+1·6	+·02	+0·7
23	− ·02	+3·3	+·04	+0·1
Aug. 6	+ ·19	+1·1	−·04	−0·1
24	+ ·14	−2·6	·00	0·0
Dec. 7	+ ·01	−1·4	−·01	0·0
8	+ ·13	−2·9	−·01	0·0
1797				
Jan. 29	+ ·05	−3·8	+·10	+0·6
31	+ ·05	−2·5	+·12	+0·8
May 5	+ ·17	−1·0	+·03	+0·2
Aug. 18	+ ·02	−4·1	+·01	−0·1
19	+ ·18	+0·2	+·01	−0·1
30	+ ·11	−2·2	+·03	−0·2
Sept. 15	+ ·07	−4·2	+·06	−0·5
18	+ ·15	−0·2	+·07	−0·6
1798				
Jan. 7	·00	−0·3	+·06	+0·1
Apr. 21	+ ·22	+2·3	+·04	+0·3
22	+ ·10	+1·7	+·05	+0·3
23	+ ·13	+3·4	+·06	+0·3
24	+ ·28	+3·9	+·06	+0·4
25	+ ·08	+1·5	+·07	+0·4
July 13	+ ·25	−0·9	−·01	0·0
15	+ ·17	−1·7	−·01	0·0
29	− ·04	−8·4	+·01	0·0
31	+ ·10	+1·6	+·01	0·0
Aug. 20	+ ·20	+0·1	+·05	−0·4
21	+ ·14	−1·5	+·05	−0·4
22	+ ·24	+1·2	+·05	−0·4
Oct. 6	+ ·16	+0·5	−·13	+1·0
12	+ ·24	−2·3	−·07	+0·6

THE EXPANSION HYPOTHESIS FOR PLANETARY NEBULÆ.

H. Zanstra, Ph.D.

1. *The Alternatives: Equilibrium without Rotation, Equilibrium with Rotation and Expansion.*—In a former paper by the author * the conclusion was reached that the absolute true bolometric magnitude of the central star of a planetary nebula is about $0^{m}.0$, independent of its temperature as given by the theory of nebular luminosity. For the extreme temperatures of 25,000° and 100,000° these correspond to radii of the central star of $\frac{1}{2}$ and of $\frac{1}{30}$ of the Sun's. From Eddington's mass-luminosity law the constant mass would be from 2 to 4 ⊙. In the same paper a doubling of the *N* line in certain planetaries, observed by Campbell and Moore,† was interpreted as an expansion of the nebular envelope, and the velocity of expansion was given. The next step would be to assume that no planetary is in a state of equilibrium, but that they all expand. This is the kind of thing one would expect if, as is generally believed, planetaries originate from novæ, where the outburst is accompanied by the ejection of a gaseous shell.

That a planetary nebula cannot be in equilibrium without rotation is clear from the Boltzmann principle. At the surface of the star the negative potential energy of an atom is so much larger than the mean kinetic energy of temperature motion, that an absurdly high concentration of matter would be required at this surface in order to yield in the outlying parts of the envelope the few atoms per c.c. necessary for nebular luminosity, and the mass of the system would become quite inconceivable. It is true that the introduction of radiation pressure would modify this conclusion, but it is believed that it would not remove the main difficulty.

In a gaseous envelope which rotates about the central star an equilibrium might be reached without necessitating a large mass. Campbell and Moore, in the publication cited above, concluded from the shape of spectrum lines that in many planetaries there is evidence of rotation for the gases. This evidence is strengthened by the fact that they usually found the rotation effect present for the major axis and absent for the minor axis. Nevertheless, it appears that shifts of spectrum lines could also be interpreted as arising from differential motions which are not necessarily rotations. In the paper mentioned above we have given examples for which this alternative interpretation, in particular that of an expanding shell, is probable. For other cases an explanation without rotation would appear somewhat artificial, but yet permissible. Evidence against the conception that a planetary is a rotating mass of gas in equilibrium might be found in their shapes which,

* H. Zanstra, *Zeitschrift für Astrophysik*, **2**, 329, 1931.

† W. W. Campbell and J. H. Moore, *Publications of the Lick Observatory*, **13**, 77, 1918.

though having some decided characteristics of symmetry, are in other respects rather irregular.*

If a planetary nebula does not rotate, it appears from the above that it cannot be in equilibrium and must therefore either expand or contract. Such an expansion or contraction of the gaseous shell would manifest itself in a doubling of the centre of the spectrum lines, to which phenomenon we have referred above. In all cases where this doubling has been clearly observed the velocity of the shell has turned out to be greater than the velocity of escape for gravity, which means that these shells must be expanding, since there is no physical cause by which they could move towards the star with such high velocities. Hence the doubling of spectrum lines is conclusive evidence that a number of planetary nebulæ at any rate cannot be in a state of mechanical equilibrium, but that they expand. In the last column of Table I this expansion is indicated by the letter E, followed by the differential velocity in km./sec., which is about twice the velocity of expansion. Of course, if the expanding gases do not all move with the same velocity, a broadening rather than a doubling of the centre of the spectrum line might result. In the following we aim at pursuing the consequences of such an expansion hypothesis, in particular as regards the statistical distribution of the radii of the envelopes and of their velocities of expansion. To this end we first wish to examine the observational material.

2. *Observed Nebular Diameters and Velocities of Expansion.*—Table I contains the nebulæ discussed in our former paper, arranged according to their real radii r_n. The first column contains the N.G.C. number of the nebula. The temperature of the central star in thousands of degrees, $T/1000°$, contained in the second column, was obtained by means of the theory of nebular luminosity. It is based on the difference in magnitude of central star and nebula, and since this difference is not known with much accuracy, the T-values are only approximate. The distance r in parsecs contained in column 3 is essentially based on the value zero for the true bolometric magnitude of the central star, which value in its turn was determined from galactic rotation.† From this distance, and the apparent radius measured from the central star to the brightest detail of the nebula (Hubble), follows the true radius of the nebula r_n given in column 4 in thousands of astronomical units. The last column contains the differential velocity in km./sec. derived from the data of Campbell and Moore.‡ For an expanding shell it is equal to twice the velocity of expansion, v_e, and is therefore termed $2v_e$. Such clear cases of expansion are indicated by the letter E, which is followed by the sign $\sim$ if the components were not clearly separated. Other differential velocities were mainly interpreted by Campbell and Moore as rotations. In adopting the expansion hypothesis, we shall treat them as expansions, and assume that their values afford an estimate of $2v_e$.

* Compare the pictures and descriptions by H. D. Curtis, *Publications of the Lick Observatory*, **13**, 57, 1918.

† The fact that later on r is also used for the distance to the central star should not cause confusion.

‡ For the Crab nebula, N.G.C. 1952, Sanford's observations were used.

TABLE I

N.G.C.	$T/1000°$	r (parsecs)	$r_n/1000$ (ast. units)	$2v_e$	N.G.C.	$T/1000°$	r (parsecs)	$r_n/1000$ (ast. units)	$2v_e$
6572	34	440	1·3	$E\sim$ 7	6751	30	2000	20	
6210	40	530	2·1	E 17	4361	37	1050	21	
I 418	36	440	3·1	n	2371,2	39	1300	22	
II 3568	38	670	3·4	n	6778	39	2500	22	
6543	40	440	3·6	E 24	6058	28	2300	23	
2392	31	410	3·7	E 106	1501	38	1100	27	
II 2149	37	1300	3·9	n	6720	70	890	27	E 38
6803	40	1550	4·2	n	7354	50	3500	28	
7009	50	350	4·2	$E\sim$ 30	6905	45	1600	29	
3242	55	340	4·4	E 21	1514	26	520	31	
7662	50	590	4·7	E 48	6804	31	1950	39	
6826	35	460	5·0	8	2452	85	5200	42	28
II 4593	22	1050	5·2	4	6853	75	470	42	
J 320	34	1600	5·6	13	2610	50	2600	46	
1535	38	580	5·8	8	2438	85	1600	48	
6891	23	1700	5·9		6445	140	2700	48	
6567	45	2000	6·0	15	246	37	500	50	
7026	45	2200	6·7	E ?	6369	50	3700	52	
6309	40	1550	7·8		6894	60	2700	58	
6818	70	920	9·2	E 42	6781	60	1200	61	
I 351	40	2300	9·3	5	1952	100	1300	64	E 2400
40	35	680	11·5		650,1	85	1650	66	
6620	35	4600	11·5		6563	80	3700	67	
2022	45	1400	15·5		7008	28	2050	82	
6629	34	2300	16		3587	55	1000	100	
6578	37	4100	16·5		7293	85	350	105	
6439	45	9500	19		6772	70	4400	115	
II 1747	34	3300	20		7139	80	3800	135	

As regards Campbell and Moore's determinations of differential velocities we remark the following: In general the nebula becomes fainter as r_n increases. The last 16 nebulæ of Table I (from N.G.C. 6853 onwards) were therefore hard to observe, the slit was generally placed on the brightest portion, and in some cases the nebula was too faint altogether. This accounts for the absence of data on differential velocities. The 19 nebulæ from N.G.C. 40 to N.G.C. 2452 were also generally underexposed (except for the ring nebula in Lyra, N.G.C. 6720, where long exposures from 16 to 24 hours were given), and, with two exceptions, no data on differential velocities were obtained.

We shall therefore do better to confine a statistical discussion of the velocities of expansion to the first 21 nebulæ, having radii smaller than 10,000 astronomical units, excluding N.G.C. 6309, for which the time of exposure was too short. They were all well investigated, but, of course, the differential velocity given means only the order of magnitude of $2v_e$. When the description of the spectrum justifies the conclusion that the lines are narrow and show little structure or inclination, we have indicated this by the letter n, and for these cases we may adopt a value $2v_e$ smaller than 10 km./sec. For N.G.C. 6891 a shallow S shape was indicated, and it is mentioned that N_1 is quite irregular, so we have not adopted any value.

Considering all this, we have for 19 cases, 9 with $2v_e$ smaller than 10 km./sec., 6 between 10 and 30 km./sec., 2 between 30 and 100 km./sec., and only one case over 100 km./sec., so that small velocities of expansion are much more probable than large velocities.

For statistics of nebular radii we use the third and fourth columns of Table I. We shall confine ourselves to a sphere of 4000 parsecs, outside of which there appear to be only very few nebulæ, and to radii, r_n, larger than 3000 astronomical units. If, with these restrictions, we plot r_n vertically against r horizontally, we see that no points occur below the line $r_n = 2{\cdot}5r$, which corresponds to an apparent nebular diameter of 5″. Indeed, this is the minimum apparent diameter of a nebula which can be well observed in Curtis's pictures, and for which star and nebula can be separated. We may therefore expect the material to be fairly complete for $r_n > 2{\cdot}5 \times 4000$, or 10,000 astronomical units, and use this material for determining the plenitude factors for $r_n < 10{,}000$ astronomical units. The last row of the following table gives the frequency of nebulæ with $r_n > 10{,}000$ for consecutive intervals of r, from which the frequency curve can be plotted. That the increase and subsequent decrease of frequency are probably real may be seen from the other rows of the table referring to various groups.

		Distance r (parsecs)			
		0–1000	1000–2000	2000–3000	3000–4000
r_n/1000 (ast. units)	10–20	1	2	1	1
	20–30	1	4	2	1
	30–140	5	5	4	3
	Total	7	11	7	5

For $r_n < 10{,}000$ the material is only complete up to $r = r_n/2{\cdot}5$, and the plenitude factors f are then obtained from the frequency curve.

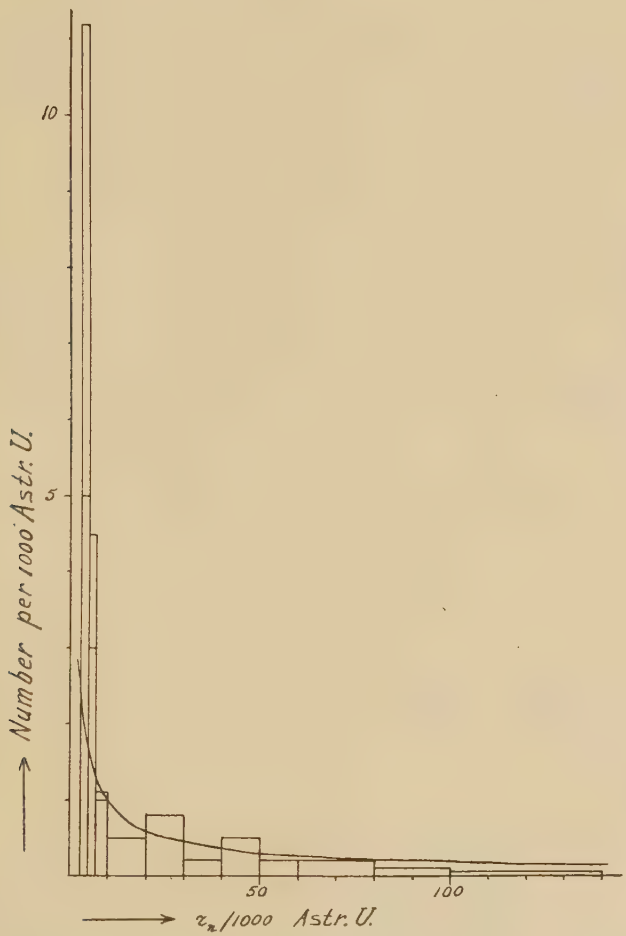
Table II contains in the first column the intervals of r_n; in the second column the number of nebulæ, n; in the third column the plenitude factor determined in the above way; and in the fourth column the number of nebulæ corrected for incompleteness, n_{corr}. The material of Table II is plotted in the figure, the area representing in each case the number of nebulæ in the given interval (for values of r_n/1000 between 0 and 10 both n and n_{corr} are indicated). Some of the intervals in the figure were taken larger than in the table.

TABLE II

All Nebulæ within a Distance of 4000 parsecs

r_n/1000 (ast. units)	n	f	n_{corr}	r_n/1000 (ast. units)	n
3–4	5	2·6	13·0	10–20	5
4–5	5	1·9	9·5	20–30	8
5–6	5	1·5	7·5	30–40	2
6–7	1	1·4	1·4	40–50	5
7–8	1	1·2	1·2	50–60	2
8–9	0	1·1	0	60–80	4
9–10	2	1·0	2·0	80–100	2
				100–140	2

The striking feature about the frequency curve is the rapid decrease of number of nebulæ with increase of nebular radius. One might object that this decrease might be due to the fact that nebulæ with large radii are faint, and therefore harder to observe. We do not think that this objection holds as long as the absorption of ultra-violet light in the envelope is complete, since then the total intrinsic brightness of nebulæ does not vary much.



Even an object with as large an r_n as the Crab nebula is a visual object and gives very good pictures with a few hours' exposure; so not many of these objects can have been missed. Neither can the decrease be due to a fall of temperature of the star as the nebula moves outward, so that the nebula would not be excited sufficiently. For if one plots the values of T in Table I against r_n, one gets a rise rather than a fall in temperature with increasing r_n. We shall therefore assume that the nebula remains visible throughout its course. For the influence of incomplete absorption see, however, section 6.

3. *General Assumptions and Approximations. Notation.*—In this section

we discuss certain assumptions and approximations underlying both sections 4 and 5.

I. We assume that, in the original star, a cataclysm of some kind occurs by which matter is ejected, so that shortly after this cataclysm, at an instant for which we put $t=0$, there is a central star of mass M and a certain amount of ejected matter. For a discussion of orders of magnitude we always take $M=3\odot$ (section 1).

II. In studying the motion of ejected matter from $t=0$ onwards, the only force taken into account is the force of gravity exerted by the central star. Hence neither light-pressure nor gas-pressure in the envelope (see section 6), nor gravity exerted by the envelope, is taken into account.

III. We make use of the following notation. At any instant t, the distance of a portion of ejected matter from the central star is indicated by q, its velocity by u. At $t=0$ we put $q=R$ and $u=V$, so that R represents the initial distance, and V the initial velocity.

For distances so large that one is justified in neglecting further influence of gravity on the magnitude of the velocity u, we have with sufficient approximation, $u=v$, the velocity at infinity. For such large distances we use the notation r instead of q, so that whenever r is used, the motion of a portion of matter may be considered uniform with velocity v.

The parabolic velocity is indicated by the suffix 0; hence, if for a portion of matter $v=0$, we have $V=V_0$ and $u=u_0$.

According to the above assumptions, the energy per unit mass E of a certain portion of ejected matter, which we follow throughout its course, is given by the law of conservation of energy

$$E=\tfrac{1}{2}V^2-\frac{CM}{R}=\tfrac{1}{2}u^2-\frac{CM}{q}=\tfrac{1}{2}v^2, \tag{1}$$

where C is the constant of gravitation.

Velocities are expressed in km./sec. throughout. If q be expressed in astronomical units, then $CM=2660$ for $M=3\odot$.

Thus (1) gives

$$3000 \text{ astr. units} \begin{cases} u_0 = 1{\cdot}34 \text{ km./sec. for } v=0. \\ u = 5{\cdot}0 \text{ km./sec. for } v=4{\cdot}8 \text{ km./sec.} \end{cases}$$

Since we study nebular envelopes of radii $\geqslant 3000$ astronomical units, the above figures show that we may take for the velocity of matter in the observed envelopes v instead of u, provided that we are satisfied with an accuracy of about 1 km./sec. In view of the fact that velocities in envelopes are presumably a few km./sec., we shall make use of this approximation.

IV. We assume therefore that the observed nebulæ with which we are dealing are in such an advanced state of development that the distances from the central star are large in the sense as formulated above, so that r increases with the constant velocity v given by equation (1),

$$r_2-r_1=v(t_2-t_1) \tag{2}$$

or, with less accuracy,

$$r=vt. \tag{2'}$$

Considering that the time for the first few thousands of astronomical units may be considerably less than follows from (2′) but never more, (2′) is also an approximation, which, however, begins to hold well for radii larger than those given by (2).

V. In making our final conclusions as to the constitution of envelopes of planetaries, we only consider matter which goes off to infinity and does not return to the star; then

$$E \geqslant 0,$$

which follows from (1), because v is real. (For the possible consideration of "returning matter" see section 6.)

4. *Only One Velocity of Ejection of a Thin Envelope.*—In this section we wish to discuss the following simple model. Let the original star eject a thin spherical envelope, all parts of which have the same outward velocity. For the initial radius R of the envelope we are to take a value of the radius of the spherical shell shortly after the cataclysm has occurred, and we may therefore assume that this value is of the same order as the radius of the original star, hence of the order of the Sun's radius. Since we have a certain freedom in choosing the instant $t=0$, there seems no objection to taking the initial radius R the same for all, or practically all, systems under consideration.

If R be expressed in solar radii $R_\odot$, the constant CM in (1) has the value 573000. To fix our ideas, let us take $R=R_\odot$. Then (1) leads to *

$$\left.\begin{array}{ll} V_0 = 1070 \text{ km./sec.} & v = 0 \\ V = 1071 \text{ km./sec.} & v = 46 \text{ km./sec.} \\ V = 1121 \text{ km./sec.} & v = 331 \text{ km./sec.} \end{array}\right\} R = R_\odot.$$

These figures only serve to illustrate that one may expect $\dfrac{V-V_0}{V_0}$ to be small as compared with unity for practically all cases we are dealing with. This quantity would, *e.g.*, be smaller than 1/1000 for final velocities v up to 46 km./sec., and the observed velocities of expansion v_e in Table I in all but two cases lie within this range.

Let, in the space under observation, the number

$$F(V)dV \tag{3}$$

of true planetaries originate per unit time (the formulation of (3) being such that the initial radius R is the same for all planetaries).

By means of (1) we transform this to another expression in which the energy interval dE occurs instead of the initial velocity interval dV. Since R is the same for all planetaries, (1) leads to $dE = d(\frac{1}{2}V^2)$. Hence, the number of planetaries originating per unit time within the energy interval dE is

$$f \,.\, dE, \tag{4}$$

where

$$f = \frac{F(V)}{V}. \tag{4a}$$

* For the sake of simplicity we really have assumed an *exact* value of 1070 for the parabolic velocity, which corresponds to the exact value $CM = 572450$. This, of course, is justified, since it may mean that our assumed mass is slightly different from $3\,\odot$.

Let the observer investigate planetaries with radii between the lower limit r_1 and the upper limit r_2. Any planetary will be in this stage of its development during a certain fraction of its life given by the time interval

$$t_2 - t_1 = \frac{r_2 - r_1}{v}, \tag{5}$$

approximately, as follows from (2). The number of planetaries of a certain kind observed at a certain instant between r_1 and r_2 is equal to the number originating in this time interval $t_2 - t_1$.

Hence, the number of planetaries within the energy interval dE which, at a certain instant, are observed to have radii between r_1 and r_2 is, according to (4), $(t_2 - t_1)f \,.\, dE$, or, according to (1), $E = \frac{1}{2}v^2$, and (5)

$$N(r_2) - N(r_1) = (r_2 - r_1)f \,.\, dv, \tag{6}$$

where

$$f = \frac{F(V)}{V}. \tag{4a}$$

For further discussion we return to the figures obtained above for V and v. Before the cataclysm the velocity of matter was zero. Now it is evident, from physical grounds, that the chance for obtaining a velocity V after the cataclysm of 1070 km./sec. cannot be very much different from the chance of obtaining a velocity of 1071 km./sec., since the mechanism by which the velocity is generated cannot have very much to do with the value of the parabolic velocity. This, however, means that for velocities of expansion v between 0 and 46 km./sec.—that is, the large majority of cases—$F(V)$ is nearly constant, and since also V varies only by about 1 part in 1000 at most, it seems legitimate to consider their ratio f to be constant.

In order to make clear just what assumptions are made as regards $F(V)$ resulting in the constancy of f, we may put the argument in a more rigorous form. For the present case this more rigorous formulation is not of very much importance, but we wish to carry it out in full, because a similar argument has to be carried out in the next section, where the situation is much more involved.

We make use of the result obtained at the beginning of this section,

$$\frac{V - V_0}{V_0} << 1, \tag{7}$$

where $<<$ means "small compared with."

Developing V and $F(V)$ into powers of $V - V_0$, we have

$$V = V_0 + V_0 \frac{V - V_0}{V_0},$$

$$F(V) = F(V_0) + \frac{1}{1!}\left(\frac{dF}{dV}\right)_0 V_0 \frac{V - V_0}{V_0} + \cdots$$

where the suffix 0 refers to the value for $V = V_0$.

Because of (7), the first order term in each expression is small compared with the main term, provided that, as regards their absolute values,

$$\left(\frac{dF}{dV}\right)_0 \text{ is not large compared with } \frac{F(V_0)}{V_0}. \tag{8}$$

The higher order terms are much less likely to influence the value of the function. Hence, it seems legitimate to look upon (8) as the condition for the approximate constancy of f.

If it be fulfilled, we retain only the main term in the expansions, so that (6), (4*a*) become, approximately,

$$N(r_2) - N(r_1) = (r_2 - r_1) f_0 \, . \, dv, \tag{9}$$

where

$$f_0 = \frac{F(V_0)}{V_0}. \tag{9a}$$

Since, because of (9*a*), f_0 is a constant independent of the initial conditions, equation (9) leads to the following conclusions :—

(*a*) Different radii of the envelope are equally probable.

(*b*) Different velocities of expansion v are equally probable for a given interval of radius.

Both these results are contrary to the facts as summarised in section 2, where it was found that nebulæ become much rarer for larger radii and for larger velocities. It is true that the observed decrease of number with increasing radius might in part be due to incomplete absorption (section 6). However, even then the result (*b*) still remains.

We therefore conclude that the simple model discussed in this section—that is, the ejection of a thin spherical envelope—is not capable of giving an insight into the outstanding facts. This model could be applicable in a few cases at most.

5. *Ejection of an Envelope with a certain Range of Initial Velocities. Comparison with Observation.*—Since the simple model of a thin, expanding envelope does not seem capable of explaining the outstanding facts, we wish to discuss in this section the more complicated model of an expanding envelope in which a certain *range* of initial velocities occurs. The part played by the velocities for a number of different nebulæ in the previous section is now largely taken by the various velocities in the same nebula, so that it will essentially suffice to consider the development of one nebula only. The probability that a certain stage of development would be observed is therefore proportional to the interval of time predicted for this stage in one nebula, and gives the statistics for nebulæ of one kind. The fact that not nebulæ of one kind but of various kinds are observed is then taken into account by a simple summation in the final results.

Apart from any definite picture we wish to make as to the origin of the range of initial velocities, it is very plausible that it should occur, if one considers that a range of 1070 to 1071 km./sec. for initial velocities at one solar radius from the centre corresponds, at large distance, to velocities of expansion between 0 and 46 km./sec. Since, for such a violent process as

the outburst of a star must be, a range of at least 1 km./sec. may easily be expected, the existence of a range of initial velocities seems nearly unavoidable, if the velocities of expansion are no higher than about 50 km./sec., as is observed for the large majority of cases (Table I).

We shall assume (in addition to all assumptions and formulæ of section 3) that a cataclysm by which matter is ejected occurs such that, shortly afterwards, a certain range of initial velocities V occurs at different distances R (of the order of one solar radius) from the centre of the system. For the sake of simplicity we assume radial symmetry, $R=R(V)$. We, moreover, assume that the velocities of different portions of matter do not interfere—in other words, that, from $t=0$ onwards, no portion of the envelope will overtake another portion.

Let the mass dM of ejected material having initial velocities between V and $V+dV$ be

$$dM=\Phi(V)dV, \tag{10}$$

$$R=R(V). \tag{10a}$$

Though we do not specify our initial picture beyond this, we remark that the condition expressed by (10), (10*a*) might well arise after the collapse of a star according to Milne's theory as to the origin of novæ.

We transform to an expression containing the energy interval dE. We have (10)

$$dM=\frac{\Phi(V)}{V}d\frac{V^2}{2}. \tag{11}$$

The equation of conservation (1) gives

$$dE=d\left(\frac{V^2}{2}-\frac{CM}{R}\right).$$

Moreover, (10*a*)

$$d\frac{CM}{R}=-\frac{1}{V}\frac{CM}{R^2}\frac{dR}{dV}d\frac{V^2}{2},$$

so that

$$dE=\left\{1+\frac{1}{V}\frac{CM}{R^2}\frac{dR}{dV}\right\}d\frac{V^2}{2}, \tag{12}$$

giving the desired transformation.

Hence, the mass dM of ejected material having energies between E and $E+dE$ becomes, according to (11) and (12),

$$dM=\phi\,.\,dE, \tag{13}$$

where

$$\phi=\Phi(V)\left\{V+\frac{CM}{R^2}\frac{dR}{dV}\right\}^{-1}. \tag{13a}$$

From (1) and (2′) we have, approximately,

$$E=\tfrac{1}{2}v^2 \quad \text{and} \quad v=\frac{r}{t}, \tag{14), (15}$$

Applying (13), which holds for a certain fixed instant t, we obtain, according to (14) and (15),

$$dM = \phi \cdot \frac{r dr}{t^2}, \tag{16}$$

so that the density of matter ρ is given by

$$\rho = \frac{\phi}{4\pi} \frac{1}{rt^2}. \tag{17}$$

Up to this point we have been considering atoms of all kinds intermixed. In order to be able to carry out a quantitative treatment, we shall now idealise this actual case, and assume that the envelope ejected consists of hydrogen, with a slight admixture of various ions emitting the nebulium lines, but not so much that it materially influences the ionisation of the hydrogen or the absorption of ultra-violet starlight. Let the *number* of hydrogen atoms (neutral and ionised) in the envelope having initial velocities between V and $V+dV$ be

$$\Phi_H(V)dV. \tag{10'}$$

Carrying out the same derivations as above with Φ_H instead of Φ we get, instead of (17), (13*a*), for the number of atoms per unit volume C,

$$C = \frac{\phi_H}{\pi} \frac{1}{rt^2}, \tag{18}$$

where

$$\phi_H = \Phi_H(V)\left\{V + \frac{CM}{R^2}\frac{dR}{dV}\right\}^{-1}. \tag{18a}$$

On account of the fact that the initial velocities occur at distances from the central star comparable with one solar radius, the value of ϕ_H does not vary much for the range of velocities concerned, provided that certain derivatives of the functions $\Phi_H(V)$ and $R(V)$ do not have extremely high values.

The *conditions for the approximate constancy of* ϕ_H are that, as regards their absolute values,

$$\left.\begin{aligned} &\left(\frac{d\Phi(V)}{dV}\right)_0 \text{ not} >> \frac{\Phi(V_0)}{V_0}, \\ &\left(\frac{dR}{dV}\right)_0 \text{ not} >> \frac{R_0}{V_0}, \qquad \left(\frac{d^2R}{dV^2}\right)_0 \text{ not} >> \frac{\left(\frac{dR}{dV}\right)_0}{V_0} \end{aligned}\right\} \tag{18b}$$

where Φ should be replaced by Φ_H. The proof of these conditions is given in section 6. By "not >>" is meant "not large compared with."

We shall assume that these conditions are fulfilled, and therefore treat ϕ_H as a constant from now onwards, meaning by this that its value is the same for different portions of matter of the expanding envelope and at different times.

Of the material in the expanding envelope only neutral atoms are capable of absorbing the ultra-violet starlight which excites the luminosity. If C_a

represents the number of un-ionised H-atoms, and C_i the number of ionised H-atoms per unit volume, the ionisation formula is *

$$C_a = Kr^2 e^{\int_0^r \alpha C_a dr} C_i^2, \tag{19}$$

with

$$K \cong \frac{4}{R_*^2} \frac{e^{\frac{157200}{T}}}{1{\cdot}222 \times 10^{15} T^{3/2}}, \tag{19a}$$

where T is the temperature of the central star, R_* its radius, and r the distance from the central star. The unit of length employed is the centimeter. The value (19a) is really a lower limit for K, and therefore gives an upper limit for the ionisation. In the following we are interested only in the order of magnitude of K, but it is essential that we assume it to be independent of r. The right-hand side of (19) then contains only the factor r^2, which is a consequence of the dilution of black-body radiation, and the exponential factor, which takes into account the diminution of intensity of the incident ultra-violet radiation through absorption by the neutral H-atoms. For the sake of simplicity only one atomic absorption coefficient α has been assumed, representing therefore a kind of average absorption coefficient for the ultra-violet starlight beyond the head of the Lyman series of hydrogen. The formula represents, of course, a somewhat idealised situation.

From (19) and (19a) it follows that for conditions usually existing in planetary nebulæ the ionisation is probably nearly complete (in our former paper two extreme examples for a nebula of radius 10,000 astronomical units gave the estimates, $C_i = 140\ C_a$ to $280\ C_a$). Hence,

$$C = C_a + C_i \cong C_i. \tag{20}$$

For the study of the absorption in the nebula one may therefore calculate C_a from (19) by putting $C_i = C$, which appears to be a fair approximation, as long as C_a/C is small compared with unity. Thus (20), (19) and (18) lead to

$$C_a \cong Kr^2 C^2 e^{\int_0^r \alpha C_a dr} = A e^{\int_0^r \alpha C_a dr}, \tag{21}$$

where A is a function of t only

$$A = K\left(\frac{\phi_H}{4\pi}\right)^2 \frac{1}{t^4}. \tag{21a}$$

Differentiating (21) with respect to r, we obtain the differential equation

$$\frac{\partial C_a}{\partial r} = \alpha C_a^2, \tag{22}$$

where the approximation sign has been omitted. The solution of this equation is, considering that, as r approaches zero, C_a approaches A (21)

$$C_a = \frac{A}{1 - \alpha A r}. \tag{23}$$

* Compare *Zeitschrift für Astrophysik*, 2, 337, 1931, where the same formula was made use of without the exponential factor.

The fraction of energy (or also the fraction of number of ultra-violet quanta) L absorbed within r at a certain instant t is therefore

$$L = 1 - e^{-\int_0^r \alpha C_a dr} = \alpha A r, \tag{24}$$

giving also the relative distribution of light within the hydrogen envelope, since the number of quanta emitted in the Balmer series is approximately equal to the number of ultra-violet quanta absorbed. Since the total energy emitted in the nebulium lines is roughly equal to the total energy of the photoelectrons generated by the ultra-violet starlight, equation (24) gives also the approximate intensity distribution of the nebulium pictures N_1 and N_2. The quantity L (24) represents therefore approximately the fraction of luminosity emitted within the radius r, under the simplified assumption, however, that there is only one absorption coefficient α for the ultra-violet starlight.

If we interpret (23) and (24) literally, we have that C_a approaches infinity and L becomes unity for $r = \frac{1}{\alpha A} = r_n$. Of course, one is only justified in applying the equations for not too large values of C_a—that is, only so long as C_a/C is small compared with unity—but one might interpret the result in this way, that for $r = r_n$ the C_a gets comparatively large, so that practically all quanta are absorbed within this radius, within which also the re-emission is confined. Hence, one might expect the luminosity to be approximately proportional to the radius r (equation (24)), until the value r_n is reached, when it drops down to practically zero.

Thus the formulæ seem to account for a rather sharp outer boundary of the luminous nebular envelope, although the nebular material generally extends quite a way beyond this luminous envelope. We are justified in using the expression "radius of the nebula r_n" instead of "luminous radius," since it actually represents the radius of the envelope as measured visually or on photographs.

This radius of the nebula r_n is given by the condition $L = 1$, and equations (24) and (21*a*) *

$$r_n = \frac{1}{\alpha A} = t^4/K_1^4, \tag{25}$$

or the age of the nebula by

$$t = K_1 r_n^{1/4}, \tag{26}$$

where

$$K_1 = K^{1/4} \alpha^{1/4} \left(\frac{\phi_H}{4\pi}\right)^{1/2}. \tag{26a}$$

The time dt for which the nebula has radii between r_n and $r_n + dr_n$ is

$$dt = \tfrac{1}{4} K_1 \frac{dr_n}{r_n^{3/4}}. \tag{27}$$

* The reader having objections to applying equations (23), (24) to the point where C_a gets infinite might define the luminous radius r_n of the nebula as the radius for which $L = \frac{1}{2}$ or some other definite fraction. Then all results obtained further remain equally valid.

This is also the number of nebulæ dN observed at any instant in the interval dr_n, if one nebula with the constant K_1 originates per unit time. Actually a number of nebulæ of various kinds originate per unit time, each of which has its own constant K_1, which depends on ϕ_H and K (equation (26*a*))—that is, on the initial distribution of material and velocities (equation (18*a*)) and on the temperature and radius of the central star (equation (19*a*)).

The number of nebulæ dN of various kinds observed within the interval dr_n is therefore, statistically,

$$\frac{dN}{dr_n} = \frac{\frac{1}{4}\Sigma K_1}{r_n^{3/4}}, \tag{28}$$

where Σ represents the summation over all nebulæ originating per unit time.

For comparison with the observational material of Table II or fig. 1, the value of the constant $\frac{1}{4}\Sigma K_1$ in (28) does not matter; the main thing is that dN/dr_n is indirectly proportional to $r_n^{3/4}$. Hence, nebulæ become much rarer as the radius increases, as is required by observation. In fig. 1 the observed dN/dr_n were plotted vertically against r_n horizontally. We have also drawn the curve (28) with the constant $10^{3/4}$. It appears to fit the observations fairly well for r_n larger than about 7000 astronomical units, but for lower values of r_n the observed number appears to be in excess by a factor 4. Thus the quantitative agreement is not good, but it is believed that this is hardly to be expected on account of the very approximate nature of the treatment.

We shall now point out some consequences as to the velocities of expansion, v_e, as measured by the Doppler effect. As such we may consider the value of the velocity of matter at a radius comparable with r_n, within which the emission of light occurs.* From (26) we have, since the escaping material has a constant velocity,

$$v_e = \frac{r_n}{t} = r_n^{3/4}/K_1. \tag{29}$$

Using (27), (29) for nebulæ between the fixed values r_n and $r_n + dr_n$, one may show that the average velocity $\bar{v}_e$ for such nebulæ is given by

$$\bar{v}_e = \frac{\Sigma v_e dt}{\Sigma dt} = r_n^{3/4}/\bar{K}_1, \tag{30}$$

where the average of K_1 refers to all nebulæ originating per unit time. Hence, this average velocity of expansion should increase with increasing nebular radius. Because of the scarcity of observed radial velocities of

* In the following, v_e is taken as the velocity of matter at $r = r_n$, being the *maximum* velocity within the luminous radius. The *average* velocity within the luminous radius, weighted according to the intensity of luminosity dL, would be one-half of this maximum velocity, and this definition of v_e is preferable if one deals with the absolute values of the constants of proportionality, but this is immaterial for the present treatment.

expansion for radii larger than 10,000 astronomical units, it is not possible at present to test this relation.

Following one nebula throughout its course, one obtains, from (26) and (29),

$$dt = \tfrac{1}{3} K_1^{4/3} \frac{dv_e}{v_e^{2/3}}. \tag{31}$$

This is also the number of nebulæ dN, if one nebula with the constant K_1 originates per unit time.

Hence, if one observes nebulæ far enough outward, the number of nebulæ in the interval dv_e is given by

$$\frac{dN}{dv_e} = \frac{\tfrac{1}{3}\Sigma K_1^{4/3}}{v_e^{2/3}}, \tag{32}$$

where Σ represents the summation over all nebulæ originating per unit time. The number of nebulæ $N(v_e)$ having velocities of expansion up to the value v_e is, therefore,

$$N(v_e) = v_e^{1/3} \Sigma K_1^{4/3}. \tag{33}$$

In section 2 we have seen that fairly complete observations of v_e are available only for nebulæ with radii smaller than 10,000 astronomical units. The following table gives in the first row the velocity intervals, in the second row the number of nebulæ observed within these intervals, and in the third row the number of nebulæ calculated according to (33).

$2v_e$ up to	10	30	100 km./sec.
Observed	9	15	17
Calculated	(9)	13	19

Now we should remark that equations (32) and (33) only hold provided one observes nebulæ of all radii, since for the derivation one has pursued each of the nebulæ through its entire course. To test (33) one really should have complete material over as large a range of r_n as possible. However this may be, it seems that the theoretical distribution of velocities of expansion agrees as regards order of magnitude with the observed, in particular that the scarcity of high velocities seems to be accounted for.

On the other hand, we must admit that, on the rather simple picture developed here, the expansion would result in a broadening of the spectral lines, whereas often an actual doubling, or at any rate a weakening of the central portion of the broadened line, is observed.

6. *Remarks. Proof of the Conditions for the Approximate Constancy of* ϕ *and* ϕ_H.—In section 4 the condition (8) for the approximate constancy of f was derived. The following proof for the conditions (18*b*) used in section 5 is quite analogous to this.

It has been assumed in section 5, where an expanding envelope with a range of initial velocities is discussed, that for the portions of matter under consideration the initial distances R from the central star are of the order

of one solar radius. For the portion of matter having the parabolic velocity V_0 this initial distance be denoted by R_0.

To fix our ideas, let R_0 be equal to one solar radius $R_\odot$, then equation (1) gives

$$\tfrac{1}{2}V_0^2 - \frac{CM}{R_\odot} = 0. \tag{I}$$

Only matter with $E \geqslant 0$ has been considered (section 3, assumption V). It has further been assumed (section 5) that velocities of different portions of matter do not interfere. For this it is necessary and sufficient that E increases with increasing R, since then, and only then, no portion of matter will ever overtake another portion (excluding the case that E is independent of R, which gives the case of a thin, expanding envelope all parts of which have the same velocity v at large distances, which was considered in section 4). From this follows that, for all material considered, $R \geqslant R_0$, or

$$R \geqslant R_\odot.$$

Combining this with $E \geqslant 0$, equation (1), section 3, gives

$$\tfrac{1}{2}v^2 = E = \tfrac{1}{2}V^2 - \frac{CM}{R} \geqslant \tfrac{1}{2}V^2 - \frac{CM}{R_\odot}. \tag{II}$$

With the numerical value $CM/R_\odot = 572450$, one obtains from (I) and (II)

$$\left.\begin{array}{ll} V_0 = 1070 \text{ km./sec.}, & v = 0 \\ V = 1071 \text{ km./sec.}, & v \geqslant 46 \text{ km./sec.} \\ V = 1131 \text{ km./sec.}, & v \geqslant 331 \text{ km./sec.} \end{array}\right\} R_0 = R_\odot,$$

so that, since the large majority of the velocities of expansion (Table II) may be expected to be smaller than 46 km./sec.,

$$\frac{V - V_0}{V_0} << 1 \tag{III}$$

for practically all cases.

Expanding the functions occurring in expression (13*a*) for ϕ into powers of $V - V_0$ and retaining only the first order term, one obtains

$$\left\{\begin{array}{l} V = V_0 + V_0\dfrac{V - V_0}{V_0}, \\ \Phi(V) = \Phi(V_0) + \dfrac{1}{1!}\left(\dfrac{d\Phi(V)}{dV}\right)_0 V_0 \dfrac{V - V_0}{V_0} + \cdots \\ \dfrac{CM}{R^2}\dfrac{dR}{dV} = \dfrac{CM}{R_0^2}\left(\dfrac{dR}{dV}\right)_0 + \dfrac{1}{1!}\left\{-\dfrac{2CM}{R_0^3}\left(\dfrac{dR}{dV}\right)_0^2 V_0 + \dfrac{CM}{R_0^2}\left(\dfrac{d^2R}{dV^2}\right)_0 V_0\right\}\dfrac{V - V_0}{V_0} + \cdots \end{array}\right. \tag{IV}$$

Because of (III), the first order terms in (IV) are all small compared with the main term, when condition (18*b*) given in the text holds. Since the higher order terms are much less likely to influence the value of the functions, one may look upon (18*b*) as a set of *sufficient conditions for the approximate constancy of* ϕ.

If they be fulfilled, then the value of ϕ is ϕ_0 given by

$$\phi_0 = \Phi(V_0)\left\{V_0 + \frac{CM}{{R_0}^2}\left(\frac{dR}{dV}\right)_0\right\}^{-1}, \qquad \text{(V)}$$

independent of V, as follows from (13*a*).

The sufficient conditions for the approximate constancy of ϕ_H and its constant value are obtained by replacing Φ by Φ_H in (18*b*) and (V).

The Rate of Increase of the Luminous Radius.—This quantity is given by $\frac{dr_n}{dt} = 4t^3/{K_1}^4$ according to (25), whereas the velocity of matter at this point is $v_e = \frac{r_n}{t} = t^3/{K_1}^4$. Hence, the rate of increase of the luminous radius is not the same as the local velocity of matter, but four times as large. If, however, the rate of increase in apparent radius is determined from filamentous details, it refers to the velocity of matter, as for the Crab nebula, the only case in which an apparent expansion has been observed.

Total Mass contained in the Envelope.—In our previous paper we estimated, by means of formulæ (19), without the exponential factor, and (19*a*), that the total amount of hydrogen, neutral and ionised, in an envelope of radius $r_n = 10{,}000$ astronomical units was of the order $0{\cdot}01\odot$, under the supposition that the envelope was a shell of thickness 2000 astronomical units, in which the absorption was fairly complete. Though at present our assumption is somewhat different, since we assume the main absorption and re-emission to occur throughout the sphere of radius r_n, we may presumably take the same estimate for the order of the mass contained within the luminous radius r_n. Assuming a velocity of expansion—that is, the velocity of luminous matter within r_n—of the order 10 km./sec. (Table I), and a maximum velocity of the order 300 km./sec. for the more outward moving matter, we have, since the amount of matter is very roughly proportional to the energy interval $\frac{1}{2}v^2$, that the total amount of matter within the envelope would be of the order of $9\odot$.* This estimate may be somewhat high, since the actual ionisation is less than by (19), (19*a*), but it seems clear that the expansion hypothesis as discussed here requires a mass in the envelope of the order of $1\odot$, or perhaps even more. That the mass of the expelled matter thus becomes comparable with that of the star from which it originates appears possible, since at the explosion an appreciable portion of the original star might well be blown away.

Radiation Pressure.—In the foregoing the influence of radiation pressure has been completely neglected. Adopting the value zero for the true bolometric magnitude of the central star (section 1), the energy emitted is $3{\cdot}28 \times 10^{35}$ erg/sec. Of this the fraction a, varying from 0·14 at $T = 25{,}000°$

* (13) and (1) give the mass $M = \int \phi dE = \int_{v_0}^{v_{\max}} \phi d\frac{v^2}{2}$. ϕ is a constant for not too high v, so that (III) is fulfilled. For high velocities ϕ may be expected to vary considerably. In the above estimate of the order of magnitude of the integral we have taken an " effective range of v " from 0 to 300, over which we consider ϕ constant and equal to its value for low velocities

to 0·9 at $T = 100{,}000°$, is absorbed in the envelope, assuming complete absorption beyond the head of the Lyman series of atomic hydrogen. Let this radiation act on a mass of $1/n$ ⊙. The velocity gained over a distance of r astronomical units under the action of radiation pressure is then found to be $4{\cdot}05 \times 10^{-3}\sqrt{nar}$, assuming that the momentum gained is energy/c. Thus, for 1/100 ⊙, which by the previous remark would be the order of the mass contained within the luminous radius, the velocities gained over 1000, 10,000 and 100,000 astronomical units are 1·28, 4·05 and 12·8 km./sec., multiplied by $\sqrt{a}$. One might conclude from this that the influence of radiation pressure, though undoubtedly present, would not change the order of magnitude of the effects calculated in section 4. The foregoing, however, implies that radiation is absorbed only once, and then re-emitted in all directions. This, it seems, can be the case only if the envelope does not completely surround the star.

Gas Pressure.—The fact that gas pressure has been neglected seems a rather serious point. There may, however, be some justification for this if one considers that, according to (24), the primary absorption of ultra-violet starlight, and therefore the ejection of photo-electrons, is practically confined to the region within the luminous radius r_n. Since gas pressure is mainly kept up by these photo-electrons, the darker outlying parts of the nebula would probably not be greatly influenced by it, nor would they have been in the past, except for a period shortly after the cataclysm. So it seems that the luminous radius overtaking the dark matter (see the second remark of this section) would find a situation not radically modified by gas pressure, but that within r_n considerable influence of pressure on the distribution of matter and of velocities might result.

Returning Matter.—One of the most serious points neglected in the previous derivation is that, if a range of velocities is assumed, the material ejected does not all go off to infinity, but that an appreciable portion has a velocity lower than the parabolic velocity and will return to the star. Provisional calculations indicate that this returning matter within the nebula forms a core, the share of which in the absorption and re-emission becomes less as time goes on, and the nebula expands. One will have to assume that the material which returns to the star is assimilated by the stellar surface. The bulk of this material, let us say 99 per cent., would not go farther away than a few astronomical units, and would return to the star during the first few years of the life of the nebula. So it seems that conditions for assimilation are not altogether unfavourable. It seems that the influence of this returning matter might well modify our results, especially for the smaller radii. All this, however, requires separate investigation.

Influence of Incomplete Absorption.—The large mass of perhaps a few Suns required for the total material to be present in the envelope (third remark) makes it likely that, as the envelope expands, the total optical thickness of all the material in the envelope finally becomes insufficient to ensure fairly complete absorption. The stage at which this occurs will, of course, depend on the mass of the material ejected; in general it should take place at high values of v_e. After the absorption gets incomplete, the

total luminosity of the envelope will soon fall off, about inversely proportionally to the square of the radius, so that one may expect that then the nebula will soon be no longer observable. Hence, the number of nebulæ observed would fall off more rapidly with the radius than is theoretically predicted. It seems possible that the discrepancy between observed and theoretical values in fig. 1 might in part be due to this cause.

Summary.—The observed fact that the emission lines of an appreciable number of planetaries are doubled or broadened at the centre shows that these nebulæ at any rate cannot be in equilibrium but must be expanding, since the radial velocity with respect to the star is larger than the parabolic velocity under gravity. An expansion of this kind may also be expected if planetary nebulæ originate from novæ, for which conception, as is well known, there is considerable evidence.

In the present paper an attempt is made to pursue the consequences of the hypothesis that a planetary originates by some kind of violent cataclysm at or below the surface of a star, by which a gaseous envelope is ejected, similar to what has been observed for certain novæ. The simplest hypothesis, that the ejected shell would have one definite velocity (section 4), leads to the conclusion that different radii of nebulæ as well as different velocities of expansion are equally probable, which is contrary to the facts (section 2), so that this hypothesis is given up.

Better agreement with observation is obtained under the rather plausible assumption that, in the ejected envelope, not one velocity but a range of velocities occurs (section 5). The number of planetaries per unit of the radius r_n is then indirectly proportional to $r_n^{3/4}$, which agrees with the fact (section 2) that planetaries of large radius are scarce. The quantitative agreement is not good. For planetaries with radii smaller than about 7000 astronomical units the theoretical curve falls considerably below the observed (fig. 1). For the velocity of expansion v_e it is found that the number of planetaries per unit of v_e is indirectly proportional to $v_e^{2/3}$, which agrees with the fact (section 2) that high velocities of expansion are rare. Because of the assumptions made, the results are only approximate, so that perhaps quantitative agreement is not to be expected.

The total mass contained in the envelope would have to be rather high, of the order 1 ⊙, or several ⊙. The possible influence of radiation pressure and of gas pressure is very briefly considered. An imperfection in the treatment is pointed out, namely, that no account is taken of the material which, after being ejected, returns to the star in course of time.

Natuurkundig Laboratorium der
Universiteit, Amsterdam:
1932 *September.*

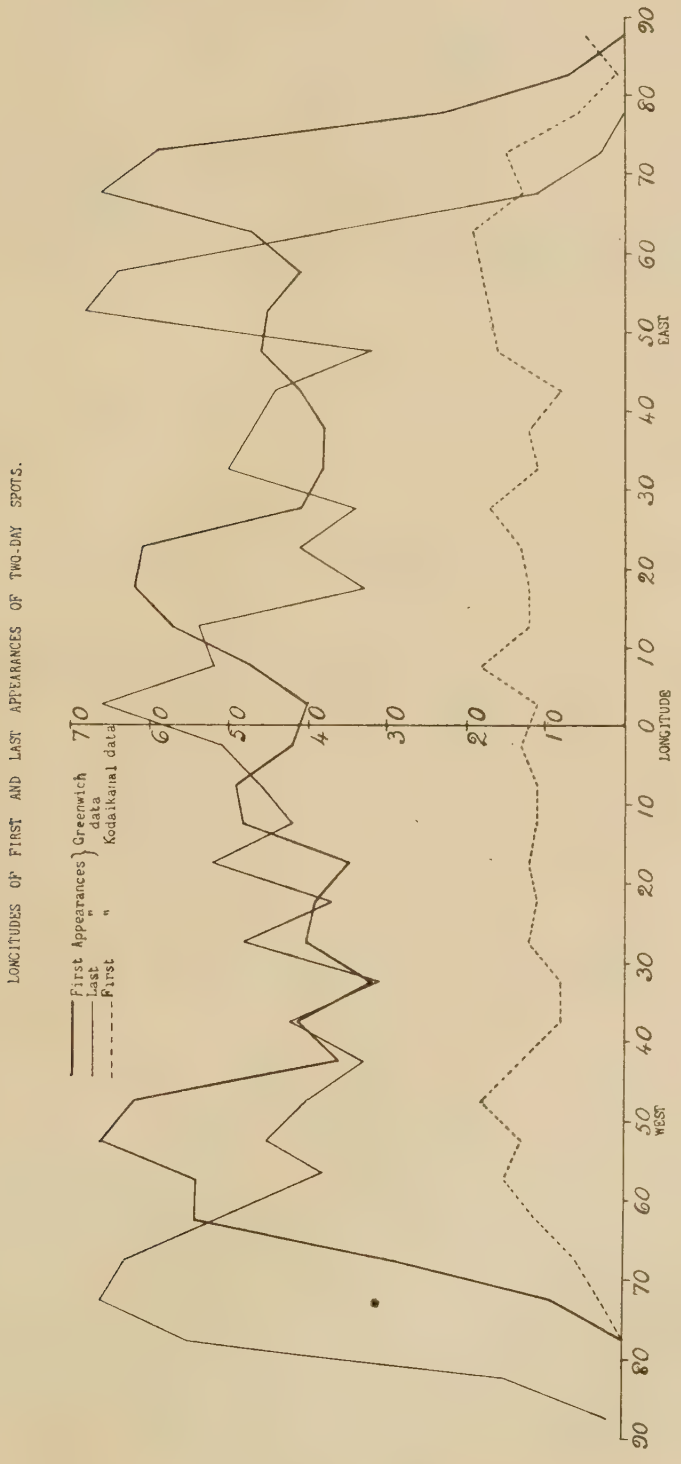
TWO LONGITUDINAL ZONES OF APPARENT INHIBITION OF SUNSPOTS ON THE SOLAR DISC.

P. R. Chidambara Aiyar, B.A.

In a paper (*Kodaikanal Observatory Bulletin*, No. 97) which is now in the press, I have given the results of a study of the occurrence of sunspots in different longitudes on the visible disc of the Sun. For the investigation, the data of sunspots were taken from the Kodaikanal records for the years 1909 to 1929. Some data from the Greenwich Photoheliographic Results were also utilised for verification of the results. By noting the longitudes in which one-day spots were observed, it was found that the frequency of their occurrence showed a minimum between longitudes 30° and 50° on either side of the Sun's central meridian. This was confirmed by a similar study of one-day spots catalogued in the Greenwich Photoheliographic Results for the years 1874 to 1912.* From the Kodaikanal records the longitudes of the first appearances of spots of longer duration were analysed and, although the same effect was seen in their cases also, it appeared to be less marked, as mentioned in the Annual Report of Kodaikanal for 1930.

A further study of the question has now been made. From the Greenwich Photoheliographic Results the "longitudes when first seen" and the "longitudes when last seen" of two-day spots (*i.e.* spots which were seen on two consecutive days) were separately analysed by entering them in columns of 5° steps. Obviously the two are not really independent, but it was thought that by considering the two phases new features, if any, in the distribution of the beginnings and endings of their existence on the visible disc might be brought out. All the two-day spots from 1874 to 1930 were considered and during the fifty-seven years there were 1395 such spots. In the accompanying diagram the abscissæ represent the longitudes, and the ordinates the numbers of the spots. The thick-line curve shows the distribution of frequency of their first appearances and the thin-line curve that of their last appearances, so that between them the two curves tell the story of the life of these spots. As the spots are carried round by the Sun's rotation during the interval between the two observations, one would naturally expect the curve of first appearances to be bodily shifted so as to be identical with the curve of last appearances, except for a slight difference due to proper motions of the spots, but in the diagram the thin-line curve differs considerably in the minor details from the thick-line one. This is due to the facts that for the purposes of the analysis the spots have been grouped together arbitrarily in 5° zones and that the two consecutive days' observations have not taken place at the same hour in all cases. It will, however, be

* The distribution of one-day spots in longitude was subsequently examined statistically by Dr. S. R. Savur, Meteorologist, Poona, and he found that the dips in the curves at the regions mentioned were real and not due to random variations.


LONGITUDES OF FIRST AND LAST APPEARANCES OF TWO-DAY SPOTS.
First Appearances } Greenwich data
Last " "
First " Kodaikanal data
70
60
50
40
30
20
10
0
LONGITUDE
10
20
30
40
50
60
70
80
90
WEST
EAST

readily seen that the phenomenon of minimum frequency of occurrence noticed above, viz. between longitudes 30° and 50° E. and W., is brought out with remarkable emphasis. Not only the births of these spots but also their deaths are comparatively rare in the two regions.

The bottom curve (dotted line) represents the distribution of the first appearances of two-day spots taken from Kodaikanal records for the period 1909 to 1929. There were only 389 spots. The troughs in the curve indicating the regions of minimum frequency are less marked and it is now evident, from a comparison of the curves, that this is due to the meagreness of the material available.

Kodaikanal Observatory :
1932 October 20.

SPIRAL ARMS AS ENVELOPES.

B. M. Peek, M.A.

In a recent paper * I discussed the formation of spiral arms by the shedding of matter from two antipodal points on the rim of a rotating and contracting lenticular nebulous mass.

Four assumptions were made, the most fundamental being that the points of ejection would always lie immediately below the matter most recently shed, the proximity of the latter giving to it a predominating effect in raising tides in the nebula. Under these circumstances the points of ejection move as though they were actually objects floating upon the rim of the nebula.

The discussion was conducted as though the rim at the points of ejection were in circular orbital motion about the centre, while the nebula itself was gradually contracting. Subsequent calculation has shown that, in the cases considered, the linear velocity of the rim and the radial velocity of contraction would have been of the same order of magnitude; the previous method of treatment could not therefore have given more than the roughest of approximations to the actual movements of the ejected particles, since these would not leave the nebula in a direction making a right angle with the radius vector, nor would the free orbits described by them be circular.

When these factors are taken into consideration, there emerges the following very significant fact, which was masked by the treatment adopted in the previous discussion.

Since the particles just ejected and the points of ejection have instantaneously velocities which are identical in magnitude and direction, the path traced out by the points of ejection is the *envelope* of the orbits described by the ejected particles.

* *M.N.*, 92, 707, 1932.

With the remaining assumptions the same as in the previous paper, let the rate of radial contraction of the rim be given by

$$-\frac{dr}{dt}=\phi(r),$$

where $\phi(r)$ is a positive function of r.

Then for every radius of the nebula there is a certain definite angular velocity of the rim, for which particles are on the point of being free to describe their own orbits under gravity, actual ejection taking place at the two antipodal points. Let this critical angular velocity be defined by

$$\frac{d\theta}{dt}=f(r).$$

If dots denote differentiation with respect to t and dashes with respect to r, then $\dot{r}$, $\ddot{r}$, $\dot{\theta}$ must all be the same for the free orbit and for the rim of the nebula, so that

$$\phi(r)\phi'(r)=r\{f(r)\}^2-\gamma Mr^{-2},$$

on applying the ordinary differential equations for an orbit, or

$$f(r)=\{r^{-1}\phi(r)\phi'(r)+\gamma Mr^{-3}\}^{\frac{1}{2}}, \qquad (1)$$

M being the mass of the nebula and γ the gravitational constant.

If the ejected particles are to leave the nebula and not immediately to fall back into it again, $\ddot{r}$ of the orbit must be more positive than $\ddot{r}$ of the rim, that is

$$(3r^{-4}h^2-2\gamma Mr^{-3})\phi(r) > -[\{\phi'(r)\}^2+\phi(r)\phi''(r)]\phi(r),$$

or

$$3r^{-1}\phi(r)\phi'(r)+\gamma Mr^{-3}+\{\phi'(r)\}^2+\phi(r)\phi''(r) > 0, \qquad (2)$$

where h is twice the orbital rate of description of areas.

This inequality is clearly satisfied by a majority of the forms that could reasonably be assumed for $\phi(r)$.

Considering now the locus of the points of ejection, we have

$$\frac{d\theta}{dr}=-\frac{f(r)}{\phi(r)},$$

whence

$$\theta=\int_r^{r_0}\frac{\{r^{-1}\phi(r)\phi'(r)+\gamma Mr^{-3}\}^{\frac{1}{2}}}{\phi(r)}dr, \qquad (3)$$

where r_0 is the radius of the nebula at which ejection commenced, so that this equation represents the envelope of the orbits described by the particles ejected from the nebula.

Professor E. W. Brown * has conceived the elegant idea that the spiral forms observed in the nebulæ may be due to the concentration of particles, revolving in orbits under gravity, along the envelope of these orbits, and has discussed certain conditions under which such an envelope will be a spiral; but his treatment involves the placing of highly artificial restrictions on the

* *Ap. J.*, 61, 1925.

distribution and motion of the matter concerned, and in effect only removes the difficulty of accounting for the spiral arms from one stage of the argument to another.

His main idea, however, is here adopted, and the suggestion put forward that equation (3), in giving the envelope of the orbits, thereby represents the distribution of matter observed in the arms. On this theory there is clearly no definite limit to the duration of the form of the spirals, such as the 10^8 years which had to be considered in my former paper. Eventually, one must suppose, mutual perturbations will alter the individual orbits of the particles so that the envelope becomes indefinite, but until this has come about the spiral distribution should continue to be observable.

The possibility may also be considered of a concentration of particles near the "aphelia" of their orbits giving rise to an observable spiral formation. In some nebulæ, *e.g.* M 81, a faint arm is traceable inside the main branch which might be due to this cause.

Since possible forms of the function $\phi(r)$ will be determined by physical considerations, which are outside the scope of this paper, I propose only to notice here the distribution given by equation (3) in one or two special cases which seem worthy of attention. If, in the first place, the velocity of contraction of the rim is always some constant fraction κ of the velocity due to a fall from infinity, we have

$$\phi(r) = \kappa\sqrt{2\gamma M}r^{-\frac{1}{2}},$$

in which case equation (3) assumes the form of the equiangular spiral

$$r = r_0 e^{-\theta\sqrt{2}\cot\alpha}, \tag{4}$$

where $\kappa = \cos\alpha$.

Secondly, when the velocity of contraction is κ times that due to a fall from a distance R from the centre,

$$\phi(r) = \kappa\sqrt{2\gamma M}\left(\frac{1}{r} - \frac{1}{R}\right)^{\frac{1}{2}},$$

whence

$$\theta = \frac{1}{\sqrt{2}}\tan\alpha\log\left(q\frac{\sqrt{R}+\sqrt{R-r}}{\sqrt{R}-\sqrt{R-r}}\right), \tag{5}$$

giving the form of the spiral, where $\cos\alpha$ is again written for κ, and q is a constant depending on the ratio of R to r_0.

It is easily seen, however, that neither of the above forms of $\phi(r)$ could hold up to the origin, for, as pointed out in the previous paper, when $\phi(r) = ar^n$ the nebula will shrink to a point in a finite time, if n be less than unity. They could not, therefore, be regarded as more than approximations to $\phi(r)$, which might be applicable to the outer portions of the arms, but which would require modification as the nucleus was approached.

I have in addition investigated by numerical integration the curves given by equation (3) for the following forms:

$$\phi(r) = \kappa\sqrt{\gamma M}r^2$$
$$\phi(r) = \kappa\sqrt{\gamma M}r$$
$$\phi(r) = \kappa\sqrt{\gamma M}\log r.$$

None of these could hold rigorously, since at large distances from the origin the velocity of contraction given is in excess of that due to a fall from infinity. However, by restricting the choice of the constant it is possible to remove the distance at which this occurs well beyond the region we are considering. This restriction of the constant has quantitatively the advantage that it is not possible by arbitrary choice to compel the curve to resemble more or less what we desire.

The curves obtained may be summarised as follows:

$\phi(r) = \kappa\sqrt{\gamma M r^2}$: For a contraction in the ratio of twelve to one there are far too many convolutions of the arms to be comparable with the observed spirals.

$\phi(r) = \kappa\sqrt{\gamma M r}$: Fewer convolutions than in the above case, but still too many for satisfactory agreement.

$\phi(r) = \kappa\sqrt{\gamma M}\log r$: Agreement with observed forms very promising, with a certain amount of latitude according to the choice of κ.

The fact must not be overlooked that although particles leave the nebula when inequality (2) is satisfied, there remains the possibility that their orbits will intersect the rim again at some other point. For this to be avoided the radius of the nebula must have become less than the perihelion distance of a particle before that particle arrives at the apse.

The condition for this is

$$\int_a^{a_0} \frac{r\,dr}{\sqrt{cr^2 + 2\gamma M r - a^4\{f(a)\}^2}} > \int_a^{a_0} \phi(r)\,dr, \tag{6}$$

where

$$c = -[\gamma M a^{-1} - \{\phi(a)\}^2 - a\phi(a)\phi'(a)],$$

$$a_0 = \frac{\sqrt{\gamma^2 M^2 + ca^4\{f(a)\}^2} - \gamma M}{c},$$

and a is the radius at which the particle was ejected. Although the left-hand side of inequality (6) is readily integrable, its value as a practical criterion is small, since it takes the form

$$L + M\cos^{-1} N,$$

L, M and N being rather complicated functions of a, so that its application would be extremely laborious. It seems not unlikely, however, that if the contraction of the nebula finally ceased with a rather abrupt deceleration, the particles which had been ejected within a certain radius would fall back into the nucleus. We should then have the appearance of spiral arms, which stopped short at some distance from the boundary of the nucleus.

Such is the case with the curved portions of the arms of the barred spirals, though there does not appear to be any way of accounting for the presence of the bar itself upon this hypothesis.

Finally, we may notice that a nebula, seen in the actual process of shedding the matter which was later to aggregate into an observable spiral distribution,

would show little resemblance to its final form; for not until some time had elapsed after the particles had been shed would they become sufficiently spread out along their orbits for the envelope to become a recognisable feature. Furthermore, since the time occupied by the actual contraction is short compared with the probable lives of the nebulæ, a very small percentage of these objects would be likely to be undergoing the process at any given time.

A classification of the spiral nebulæ could not, therefore, be expected to show any trend in the direction of progressive evolution.

A DISCUSSION OF METEOR ORBITS CONNECTED WITH THE PONS-WINNECKE COMET.

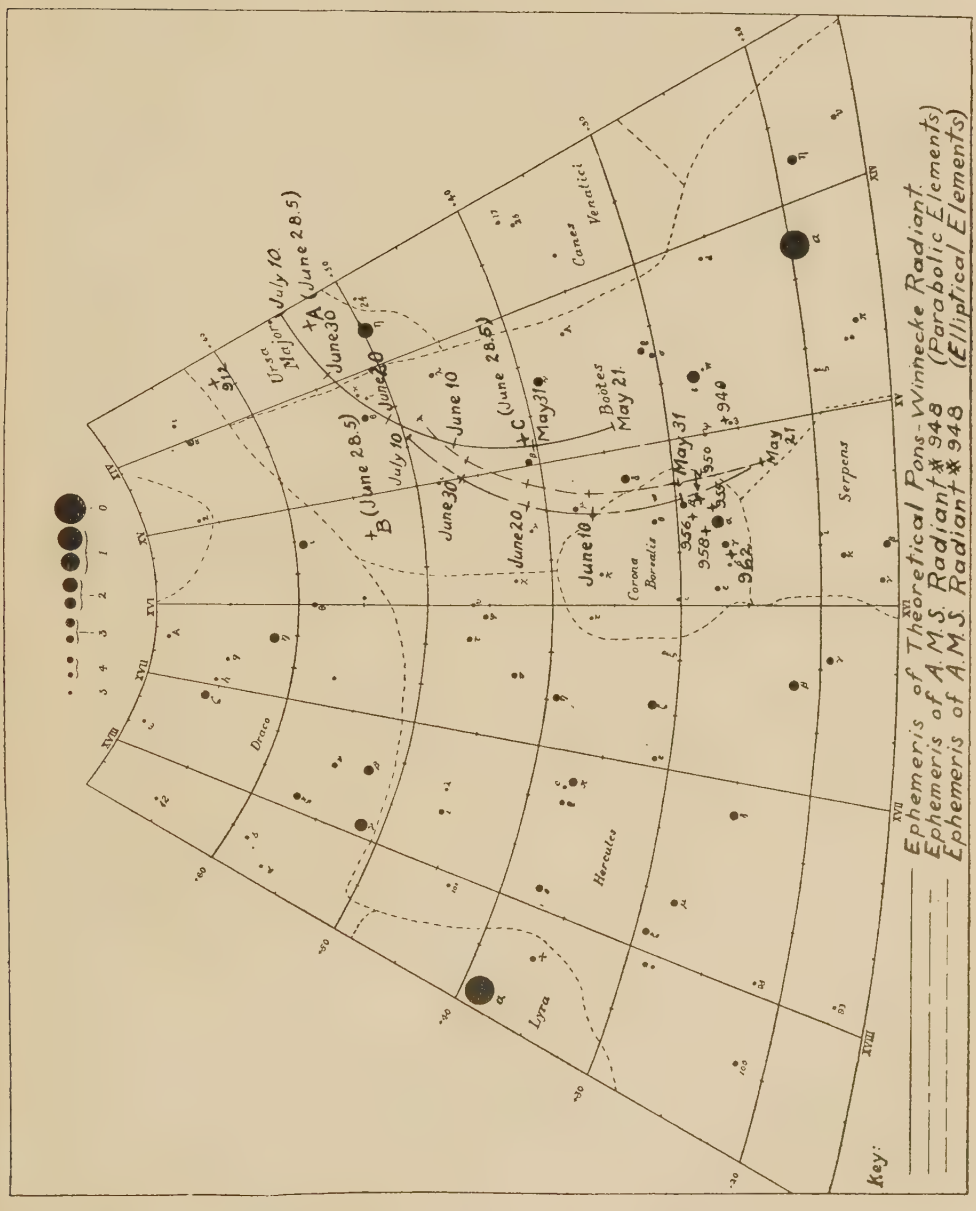
Franklin W. Smith.

(Communicated by C. P. Olivier)

In the latter part of May and in early June 1916, meteors were unexpectedly numerous, and several well-determined radiants in Corona Borealis were obtained by two members of the American Meteor Society, John Koep and Philip Trudelle. Orbits computed for these radiants by C. P. Olivier * were found to bear a close resemblance to the orbit of the Pons-Winnecke comet. On the basis of this similarity, he announced their probable connection. On June 28 of the same year the late W. F. Denning and another English observer obtained three radiants (A, B and C on figure) in the vicinity of Ursa Major. Orbits computed for these radiants also closely resembled the Pons-Winnecke orbit, a fact which Denning in turn announced.† Since we have here two separate groups of radiants appearing at different times and in different parts of the sky, and yet having similar orbits, Dr. Olivier suggested the desirability of tracing their mutual connection.

Ephemerides were computed for a typical radiant of the first group,‡ first, on the assumption that the orbit was parabolic, and, second, that it was elliptical, with the semi-major axis equal to that of the Pons-Winnecke orbit. As this radiant was very near the zenith when determined by observations, the correction for zenith attraction was neglected. In computing these ephemerides Ω was varied so as to obtain an intersection of the meteor orbit with that of the Earth at each ephemeris epoch; and in order to keep the orientation of the meteor orbit as nearly constant in space as possible, ω was varied in a compensatory manner and i was held constant. An ephemeris for a theoretical radiant corresponding to the Pons-Winnecke

* *M.N.*, **77**, 71, 1916; and *Pub. Leander McCormick Obs.*, **2**, pt. 7, 209.
† *Observatory*, **40**, 95, 1917; *Nature*, **97**, 388 and 457, 1916.
‡ A.M.S. Radiant No. 948. See *Pub. Leander McCormick Obs.*, **2**, pt. 7, 210.


Ursæ Majoris
July 10
(June 28.5)
+A
June 30
June 20
June 10
+B (June 28.5)
July 10
June 30
June 20
+C (June 28.5)
May 31
June 10
Canes Venatici
Bootes
May 21
May 31
May 21
June 10
June 20
June 30
Corona Borealis
Serpens
Draco
Hercules
Lyra
Key:
Ephemeris of Theoretical Pons-Winnecke Radiant
Ephemeris of A.M.S. Radiant * 948 (Parabolic Elements)
Ephemeris of A.M.S. Radiant * 948 (Elliptical Elements)

orbit was computed in a similar manner, and the three ephemerides (given in the table) were plotted in the accompanying figure.*

It is apparent from the figure that the radiants of the first group, *i.e.* the A.M.S. radiants, would have drifted with time toward the position of the radiants of the second group, *i.e.* the English radiants, and toward the theoretical position of a Pons-Winnecke radiant. This confirms the conclusion that the meteors from the A.M.S. radiants belonged to the same system as those from the radiants of the second group, and that their orbits were closely related to that of the comet. However, since the agreement is not perfect, and since there is an outstanding difference of about 30° in node, it is apparent that the meteors had been separated from the actual orbit of the comet by dispersive forces, which would be expected in this case because of the direct motion of the comet, its short period, and the low inclination of its orbit. It may be noticed in this connection that the perihelion distance of the comet has been increasing, so that the perihelion point, which was within the Earth's orbit in 1916, is now outside of the Earth's orbit. If the meteoric matter was previously distributed about the comet's orbit, it is obvious that the perturbations which caused this change could not have caused exactly the same changes in the orbits of the meteors, since the latter would not have been at the same distance from the perturbing body. This would account for the difference which has been noted between the meteor orbits and that of the comet. As is well known, there is some reason to think that the Siberian meteorites of 1908 were originally a part of the Pons-Winnecke system, and were separated in this way.

1916	Ω	Ephemeris of A.M.S. Radiant 948 (parabolic elements) $\Omega+\omega=283°\ 24'$		Ephemeris of A.M.S. Radiant 948 (elliptical elements) $\Omega+\omega=287°\ 36'$		Ephemeris of Pons-Winnecke Radiant $\Omega+\omega=271°\ 36'$	
		α	δ	α	δ	α	δ
May 21	60° 05′	229° 04′	25° 50′	227° 55′	22° 48′	222° 39′	33° 52′
31	69° 41′	231° 35′	29° 38′	228° 56′	29° 43′	222° 49′	40° 19′
June 10	79° 15′	232° 30′	35° 03′	228° 45′	35° 57′	221° 00′	45° 58′
20	88° 48′	231° 11′	40° 34′	226° 55′	41° 18′	216° 41′	50° 37′
30	98° 20′	227° 07′	45° 36′	222° 43′	45° 37′	209° 22′	53° 24′
July 10	107° 52′	220° 08′	49° 13′	216° 53′	48° 11′	200° 31′	54° 24′

* The method of computing was derived from *Meteors*, by C. P. Olivier, 1925. Computation was carried to four places.

ON THE ORIGIN OF THE SOLAR SYSTEM.

Friedrich Nölke, Ph.D.

(Communicated by the Secretaries)

In a paper * entitled "On the Origin of the Solar System" Dr. H. Jeffreys refers to an article in the *Handbuch der Geophysik*, where I have reviewed the principal hypotheses dealing with the origin of the solar system. As Dr. Jeffreys considers some arguments put forth by me against his own hypothesis as not being valid, because they are based on misunderstandings, I may be permitted to make some short remarks with regard to his statements.

1. On the fact that according to the tidal and collision theory only a few stars of the Galaxy can have planetary attendants, no argument of decisive character can be based. Nevertheless this fact will produce much scepticism against the hypothesis.

2. By supposing the density of the medium to be homogeneous and the time factors equal, Jeffreys finds that the excentricity of Jupiter decreases at least nine times as fast as that of Mercury. But the medium shrinks slowly from the interplanetary region and joins the Sun for the most part. In approaching the Sun the density of the contracting medium increases. Mercury in the neighbourhood of the Sun is affected, therefore, more strongly and for a longer time than Jupiter at a greater distance from the Sun.

3. Jeffreys was for a long time an adherent of the tidal theory. Even in the second edition of *The Earth* he stands to this theory and supposes the planets to have got their rotation by satellite-masses joining the planets. The difficulties adhering to this explanation then induced him to abandon the tidal theory and to replace it by the collision theory. For that reason the statement that he has "recently" modified his explanation of rotation seems to me quite correct.

My having, even now, once called Jeffreys' theory "Tidal Theory" is an error that I hope the reader will excuse.

4. The small excentricities of most of the satellites and the small inclinations to the equator of the planets are deduced by Jeffreys from the influence of a medium, but he does not define its constitution. As a planetary atmosphere extending to a great height would be such a medium producing the supposed effect, and as Jeffreys himself points to the possibility that in the first stage of development the planets might have expanded to radii comparable with the distance of many of their satellites, I presumed that he had identified the medium with the planetary atmosphere. In this case the satellites would have been exposed to resistance for only a few years.

Jeffreys now supposes that the actual motion of the interplanetary medium near the planet might approximate to a variational orbit, in which case the effect on the mean distance of satellites would be rather feeble. It would

* *M.N.*, 92, 887, 1932.

probably be difficult to prove this supposition and to demonstrate how the orbits of the satellites might have come into the plane of the planetary equator.

5. In a former calculation * Jeffreys gives 3000° for the surface-temperature of the earth. I referred to this statement in the description of the pre-geological development of the earth. The number does not, as Jeffreys believes, refer to the surface-temperature of the Sun.

6. Jeffreys refers to a conclusion of E. W. Brown that the excentricity of the earth might have been in the beginning very considerable and later on, merely by gravitational influences, reduced to the present value. It is obvious that this possibility exists ; but it would be an exception. The probability of all large planets having diminished their originally considerable excentricity by pure mutual attraction is insignificant.

7. Jeffreys supposes the mass of the interplanetary medium to be comparable with the mass of Jupiter. But in reality it must by far exceed the mass of all planets, if these are to be influenced in the way he supposes. The medium has an influence on the planets other than that of the air on a pendulum. The pendulum pushes aside the surrounding air ; but the planet holds fast by its attraction the material striking against it. Only a few particles of the planetary atmosphere, which at the moment of impact acquire a great velocity, may escape the planet. The mass of the planet, therefore, enlarges continually. Upon the increase of mass depends the decrease of excentricity ; the latter is almost inversely proportional to the planetary mass.† If the large planets have got their small excentricity in a rotating medium, almost their whole mass must have come from the medium. But as they are capable of catching only a fraction of the mass of the medium, a very large residue joining the Sun must give it a rotational moment surpassing very much the real one.

If it be the case that the planets should have got nearly all their mass from a gaseous medium, the earth too would have done so. But only a small part of the earth is gaseous. The supposition that the captured gases, by changing their state of aggregation, were transformed into solid matter, is not admissible, since condensible gases had abandoned the gas-character already in the interplanetary space and had become meteoric dust.

Finally, the rotational movement of the planets would be essentially modified by the mass captured from the medium, and it would not be admissible to reduce the present condition to shearing forces at the time of the collision of the star with the Sun.

8. My critical analysis of the possibilities of development of our planetary system leads to the result that the system has come forth from a widely spread cosmic nebula having shape and structure and splitting up into several parts, which conglomerated and formed the Sun and the planets. Perhaps the nebula may be compared with one filament of the cirrus nebulæ in the constellation of Cygnus. I do not trace back the contraction of the system

* *M.N.*, 89, 736, 1929.

† Fr. Nölke, *Der Entwicklungsgang unseres Planetensystems*, Berlin und Bonn, p. 90, 1930.

to an enlargement of the mass of the Sun, but to a decrease of the pressure of radiation, which nearly compensated the gravitation in the nebula, but gradually decreased when the masses contracted.

Favourable conditions of development exist for the regular satellites in the interior (not at the boundary) of the planetary atmospheres, which, at high temperatures and extending beyond the present orbits of the satellites, together with the planetary core, formed a whirl, the masses of which described free circular orbits in a defined ring-area of the equatorial plane.

Bremen, Germany.